Risk-Corrected Probabilities of a Binary Event

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Abstract

We obtain risk-neutral probabilities of the Brexit referendum using data from both the options and prediction markets. We then provide a risk-corrected measure of these probabilities using both non-parametric and parametric methods. While former correction marginally changes the risk-neutral probability, the effect of the latter depends on relative wealth calibration and risk preferences. We estimate subjective Brexit probabilities from past opinion polls and provide daily estimates of voting intention to leave from the BES survey. By comparing the subjective probabilities with our risk-corrected measures, our results show that both FX option and prediction market participants are likely to reveal moderate risk seeking preferences before the Brexit referendum.

Key Words: option prices, prediction markets, risk correction.

JEL Codes: F31; F37; D84; G14.

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1 Introduction

Knowledge of the ex ante physical probabilities of future infrequent but extreme events is crucial to asset pricing. These events may significantly affect asset prices, for instance, states of the world with large negative payoffs can distort prices even when their probabilities are very small. Under the "rare disasters" view, the change of event-related asset return is due to the event risk, which carries a significant risk premia (Barro, 2009; Farhi and Gabaix, 2015; Seo and Wachter, 2019). However, under the "peso problem" view, the return change of any asset is due to an expectation of a discrete shift in the return distribution (Rogoff, 1977, 1980; Lewis, 2016). Specifically, before the event outcomes' occurrence, financial market participants may speculate the potential outcomes via the event-related assets. The speculative activities and market participants' belief of future outcome have impacts on the shape of forward-looking return distributions. The degree of which the prices absorb the impact of the event before the event's actual occurrence depends on the extent market participants anticipate the event's actual outcome.

Building upon the latter standpoint, our work aims at retrieving, from market prices (options and prediction markets) and past opinion polls, a proxy for physical probability of a particular binary political event: the European Union (EU) withdrawal referendum held by the United Kingdom (UK) on the 23 June 2016 (Brexit). This event represented the possibility of a significant change in the country's international trade and immigration policy. The UK would cease to participate in that free trade area, becoming a more closed economy, at least in the short run. Agents would thus be expecting that the sterling (British pound or GBP) value to the other currencies, especially to the United States dollar (USD) would be negatively impacted with potentially severe welfare implications (Van Reenen, 2016; Steinberg, 2019).[1] Figure 1a shows that there was a sharp depreciation of the GBP to the USD on the day after the referendum. On the same day, the change in out-of-the-money put option prices was also extreme relative to normal daily changes in option prices (Figure 1b). Does this suggest that the Brexit probability was negligible or substantially underestimated in the foreign exchange (FX) spot and option markets?

To answer this question, we first investigate the risk-neutral density (RND) extracted from

[1]While simulations from GE models suggest up to 10% welfare losses for UK citizens (Van Reenen, 2016; Steinberg, 2019), empirical studies document adverse effects for British economy in terms of output loss (Born et al., 2019), foreign investment (McGrattan and Waddle, 2020) and reduction in stock valuations of FTSE 350 (Davies and Studnicka, 2018) as well as international firms exposed to Brexit (Hassan et al., 2021).

British Pound options. We find that RNDs from short maturities options have a bimodal distribution one week before the referendum, in line with the evidence provided by Kostakis et al. (2020), and risk-neutral tail risk of GBP/USD deviates from its normal level since the beginning of 2016. Therefore, it is reasonable to suppose that agents attached a probability that a discrete change in the economic fundamentals that govern the dynamics of the GBP exchange rate would occur, which is the typical description of a "Peso Problem".

Under the risk-neutral assumption, several proxies of the Brexit probability were created in the financial market, including prediction and option markets. The reciprocal of odds from prediction markets are common as probabilities of event outcomes under risk neutrality (for example, Belke et al. (2018) and Hanke et al. (2018)). More recently, Kostakis et al. (2020) estimate Brexit probabilities from the British Pounds options market, following the model-based methodology of Borochin and Golec (2016). They argue that option markets provide a good alternative to prediction markets for event probability extraction, since only some major events are considered in prediction markets. Following the literature, we also extract daily risk-neutral Brexit probabilities from both prediction and option markets, but different from Kostakis et al. (2020), we use both model-based (Borochin and Golec, 2016) and model-free (Langer and Lemoine, 2020) methods to obtain option-implied probabilities. We find that cheap out-of-the-money options carry more information about the probability of the 'Brexit' outcome. On average, the options market reveals a higher "Leave" probability by comparison to the prediction market. However, the question remains whether the attached probability implied in the financial markets is close to the physical probability of the peso event?

In this case, better estimates of physical probabilities are essential to understanding how well the financial markets anticipate the probability of binary political events such as Brexit, even though estimation of the physical probability of any event is very challenging. Sayers (2016) claims that internet polls were the closest to the Brexit physical probabilities. The underlying reasoning is that intention to vote leave was around 50% during most of the sample period, such that the actual result was within the margin of error. While Cesar et al. (2017) document a correlation between telephone and internet polls, they also show that the voting intention (constructed using 23 million tweeter hashtags) is not higher than 71%. Assuming that polls are independent referendum experiments and the market participants form a belief about the referendum outcome by learning from available polls' outcomes, we construct subjective physical probability proxy by considering a set of telephone and internet polls. On one hand, we find that past political opinion poll results capture well how market participants *learn* about the "Leave" probability. On the other hand, our findings also reveal that the probability estimates under risk neutrality from both option and prediction markets fail to match the likelihood of a Brexit outcome suggested by political opinion polls data.

The limitations of the risk-neutral Brexit probabilities extracted from prediction and options markets may cause it to deviate from the subjective probabilities from polls. One limitation is that state prices reflect agents' perception of wealth in both states of the world: with and without Brexit. Another limitation is that participants in both of these markets may have different attitudes towards risk. To eliminate these limitations, our paper thus provides both non-parametric (using the Ross-Recovery theorem - Ross (2015)) and parametric corrections (calibrating stochastic discount factors) of the risk-neutral probabilities obtained from these two markets. We argue that this risk correction is important in bringing probabilities closer to their perceived (subjective) physical measure estimated from past polls. To minimize the distance between risk-corrected probabilities and subjective probabilities, under the assumption that the agents in both markets have same expectation on the relative wealth between these two states, the average agent in prediction markets has always a stronger risk preference than the average agent in option markets. Specifically, average agents in both markets are risk seeking if they expect lower wealth in the "Leave" state compared to the "Remain" state.

Although polls reveal the proportion of "remain" and "leave" votes, there are still the crucial indecisive poll participants. Therefore, extracting physical probabilities of the future state of the world from polls could be more informative by attributing a probability that the indecisive will vote either in favor or against Brexit. This can only be reasonably done if the disaggregated Survey data is available, in addition to at least some personal characteristics of the respondents.[2] In this paper, we use the British Election Study (BES) survey data with individual-level information as an example and obtain the intention to vote "leave" as 51.07%. Daily estimates from the Survey data suggest that the survey-based estimates are less sensitive to the new information about the event compared to prediction and financial markets. Or put differently, the markets react to news (e.g., Murder of MP Cox on 16 June 2016) about a binary event outcome, while the voting intention is primarily driven by persistent characteristics (e.g., age, income, education, views about immigration and risk

[2] Venturi et al. (2021), for instance, attribute a probability, by estimating a Probit model, that the indecisive deputy would vote in favor of the impeachment of the Brazilian President in 2016.

preferences) of the voters.

The remainder of the paper is organized as follows. Section 2 shows the methodology employed to retrieve risk neutral probabilities from the option markets and the theory underlying the correction for risk. Section 3 explains the data we used in this paper. Section 4 recovers risk-neutral probabilities for the Brexit referendum, probabilities with risk-correction, subjective probabilities calculated from opinion polls and voting intention estimated from the BES Survey data. In the last section, we present our concluding remarks.

2 Methodology

2.1 Option-Implied Risk-Neutral Distributions

The risk-neutral probability distribution of financial assets contains information about investors' beliefs about the future performance of the underlying asset, and reflects investors' attitudes towards risk. Benefiting from options with a wide range of strike prices, we can use option prices to derive the risk-neutral probability distribution of the underlying asset.

To extract the risk-neutral probability distribution (RND) of the underlying asset prices, we follow the non-parametric method proposed by Figlewski (2009). This model-free method with flexible extreme value tails allows for some non-standard features in approximated RNDs, like fat tails, bimodality and so on. It is particularly useful in a binary world, like the Brexit period. We believe that this method is sufficient to infer a well-performing RND function from European options. However, the options we use in this paper are currency options listed on the Chicago Mercantile Exchange (CME), which are American options prior to 2017. Therefore, to make our data fit into this non-parametric methodology, we convert American option prices into European option prices in the RND extraction steps. The steps about how to extract a well-behaved RND is in Appendix 5.

2.2 Forward-Looking Tail Risk

Apart from analysing the moments from RNDs we are also interested in exploring the information from option-implied measures of tail risk (Ait-Sahalia and Lo, 2000; Leiss and Nax, 2018). Tail risk measures under the risk-neutral probability capture the investors' forward-looking assessment of the likelihood of the adverse market state, the probability of the 'tail event' occurring, and the evolution of their belief over time.

In order to capture changes in market expectations, we focus on changes in the quantiles of RNDs (Bevilacqua et al., 2021). We denote the $\alpha\%$ quantiles of the RND extracted from prices of contracts expiring on time T_o as $\Pi^{\leftarrow}_{t,T_o}$ and define it as

$$\Pi^{\leftarrow}_{t,T_o}(x) := R^* \;\text{ where }\; \Pi^{\leftarrow}_{t,T_o}(R^*) := Prob(R_{t,T_o} \leq \; R^*) = \alpha, \tag{1}$$

where $R_{t,T_o} = \ln \frac{S_{T_o}}{S_t}$ and S_t is the FX spot rate at time t (GBP/USD in our case). This measure is similar as option-implied Value-at-Risk (VaR) and the difference is these two measures have the opposite sign.

However, we cannot extract the RND of FX spot rate directly, since CME monthly currency options are written on the nearest futures contracts with March-quarterly maturity.[3] Thus, we make adjustment for the difference between the option and its underlying future expiry days (Cincibuch, 2004; Huchede and Wang, 2020). The price of a CME call option at time t is $C(t, K, T_f, T_o)$, with strike price K, maturity T_o and underlying future contract that expires at T_f. The time t price of a CME future with maturity T_f is F_{t,T_f}. The aim of this transform is to an option written on a future that also expires on the option expiry day, which is equivalent to an option on spot. We denote the US and UK interest rates as $r^{US}_{T_o,T_f}$ and $r^{UK}_{T_o,T_f}$, respectively. According to the no-arbitrage relationship,

$$F_{T_o,T_f} = F_{T_o,T_o} e^{(r^{US}_{T_o,T_f} - r^{UK}_{T_o,T_f})(T_f - T_o)}. \tag{2}$$

Then, we can rewrite the call option price as

$$\begin{aligned} C(t, K, T_f, T_o) &= e^{-r_t(T_o - t)} \mathbb{E}_t[\max(F_{T_o,T_f} - K, 0)], \\ &= e^{-r_t(T_o - t)} \mathbb{E}_t[\max(F_{T_o,T_o} e^{(r^{US}_{T_o,T_f} - r^{UK}_{T_o,T_f})(T_f - T_o)} - K, 0)], \\ &= e^{(r^{US}_{T_o,T_f} - r^{UK}_{T_o,T_f})(T_f - T_o)} e^{-r_t(T_o - t)} \mathbb{E}_t[\max(F_{T_o,T_o} - \bar{K}, 0)], \\ &= e^{(r^{US}_{T_o,T_f} - r^{UK}_{T_o,T_f})(T_f - T_o)} e^{-r_t(T_o - t)} \mathbb{E}_t[\max(S_{T_o} - \bar{K}, 0)], \\ &= e^{(r^{US}_{T_o,T_f} - r^{UK}_{T_o,T_f})(T_f - T_o)} C(t, \bar{K}, T_o), \end{aligned} \tag{3}$$

[3] For example, an option expiring on January is written on a future expiring on the nearest March. Even for an option expiring in March, the difference between the option expiry day and the underlying future expiry day is about two weeks.

where

$$\bar{K} = Ke^{-(r^{US}_{T_o,T_f} - r^{UK}_{T_o,T_f})(T_f - T_o)}. \tag{4}$$

Similarly, we can convert the put price from the option on the future to an option on the spot. Then, we can extract RNDs of FX spot rates from spot option prices and calculate market expectations in different horizons.

2.3 Option-Implied Risk-Neutral Event Probability

2.3.1 Model-Based Estimation

To analyze the evolution of the uncertainty about Brexit outcome, we infer a time-series of pre-Brexit probabilities by following the methodology of Borochin and Golec (2016). This methodology uses options to estimate risk-neutral probabilities and to identify state-contingent underlying asset prices and volatilities. Following the same framework, our world is binary with two limiting future states: one in which the UK leaves the European Union ("Leave" state) and the other in which the UK remains in the European Union ("Remain" state). Then, using the information from futures and options markets, we estimate five latent variables, which correspond to the futures prices and volatilities in each state of the world and the risk-neutral probability in one of the states.

Consider that the USD price of one unit of a GBP in a future contract at time t is given by F^L_{t,T_f} in the "Leave" state, and F^R_{t,T_f} in the "Remain" state, where T_f is the maturity of the future contract and t is a time point before the Brexit referendum, which was held on 23 June, 2016. In the absence of arbitrage, the future price observed at time t must be the probability weighted average of the future price in the "Leave" state and the future price in the "Remain" state:

$$F_{t,T_f} = p^L_t \times F^L_{t,T_f} + (1 - p^L_t) \times F^R_{t,T_f}, \tag{5}$$

where p^L_t is the time t risk-neutral probability that the UK leaves the European Union ("Leave" probability).

Consider a set of options on GBPUSD futures contracts with the expiry day T_o, where T_o is a time point after the Brexit referendum. In the absence of arbitrage, the European options price observed at time t ($O(F_{t,T_f}, K, \sigma_t, T_o)$) should be the probability weighted average of the theoretical option price in the "Leave" state ($\hat{O}(F^L_{t,T_f}, K, \sigma^L_t, T_o)$) and the theoretical option price in the "Remain" state ($\hat{O}(F^R_{t,T_f}, K, \sigma^R_t, T_o)$):

$$O(F_{t,T_f}, K, \sigma_t, T_o) = p^L_t \times \hat{O}(F^L_{t,T_f}, K, \sigma^L_t, T_o) + (1 - p^L_t) \times \hat{O}(F^R_{t,T_f}, K, \sigma^R_t, T_o), \tag{6}$$

where K is the strike price, σ^L_t and σ^R_t are return volatilities in the "Leave" state and "Remain" state, respectively.

We use the Black model (Black, 1976) to calculate theoretical prices for European futures options. Since the option contracts in our sample are American-style, to fit our data into equation (6), we convert the American option prices into European option prices by following the same steps in Appendix 5. First, we obtain the implied volatility from the American option price by using Barone-Adesi-Whaley (BAW) American futures option pricing model (Barone-Adesi and Whaley, 1987). Then, we use the BAW implied volatility in the Black model (Black, 1976) to calculate the price of the corresponding European option.

According to equation (5) and (6), our aim now is to estimate the five parameters at each point in time t, $\Theta_t = \left\{p^L_t, F^L_{t,T_f}, F^R_{t,T_f}, \sigma^L_t, \sigma^R_t\right\}$. We proceed by constructing a system that includes a GBPUSD future contract and N traded options that are written on this future contract with a single expiry day T_o and a wide range of strike prices K_i, where $i = 1, 2, ..., N$.

We believe that both call and put options could bring information to the system. Borochin and Golec (2016) set $N = 8$ and select eight call options only to estimate these latent parameters from the system. They argue that put option price is less reliable to be incorporated in the system since the trade of stock put options is usually less frequent than stock call options. Note that a call on GBP/USD is a put on USD/GBP vice versa, this is not necessarily true for currency options. When we look at the open positions of options, the open interest of call options is significantly lower than the put options in our sample period, which guarantees the reliability and information content of put option prices. Also, the probability of Brexit may significantly reduce the British Pound's future price, so using put options in the system to recover the Brexit outcome probability is very useful (Langer and Lemoine, 2020). Due to the reasons mentioned above, our system incorporates both put and call options with positive open interest.

Different from Borochin and Golec (2016)'s method with ATM options only, we use all observed (deep) out-of-the-money (OTM) and at-the-money (ATM) options. We believe that options with different strike prices could bring additional information to the system, for example, cheap out-of-the-money options might carry important information about tail risk (Kelly and Jiang, 2014). Following Borochin and Golec (2016), we use options with short maturity (maturity date is still after the event day), since their prices are more sensitive to changes in underlying asset price.

We denote observed asset prices in the left-hand side of equations (5) and (6) as M_t, and theoretical asset price in the right-hand side as $\hat{M}_t$. The non-linear least squares estimation of Θ_t involves numerically solving of a optimization problem with objective function

$$\Theta_t = \arg\underset{\Theta_t}{min} \sum_{i=1}^{N+1} \omega_{t,i} \{M_{t,K_i} - \hat{M}_{t,K_i}(\Theta_t)\}^2, \tag{7}$$

and constraints

$$s.t. \begin{cases} 0 < F^L_{t,T_f} \leqslant F_{t,T_f} \leqslant F^R_{t,T_f}, \\ \sigma^R_t, \sigma^L_t > 0, \\ 0 < p^L_t < 1, \end{cases} \tag{8}$$

where $\omega_{t,i}$ is the weight on observation i at time t. Our baseline specification sets $\omega_{t,i}$ as a constant value 1 across i, a standard approach used in the literature. However, cheap out-of-the-money options might carry important information about potential extreme events, so, different from Kostakis et al. (2020), we also estimate the system by assigning a large weight to error terms for cheap options. Specifically, our second and third specifications set $\omega^2_{t,i} = \frac{1}{M_{t,K_i}}$ and $\omega^3_{t,i} = \frac{1}{M^2_{t,K_i}}$, respectively (Carvalho and Guimaraes, 2018).

2.3.2 Model-Free Estimation

Model-based estimation of option-implied probabilities provides valuable insights about the time-varying event probabilities. However, this method requires an option pricing model that is able to reconcile observed and theoretical option prices. The strict assumptions of the underlying return process in option pricing models are commonly violated in the real world, e.g., the implied volatility smiles and smirks. In order to estimate the probability of an event using a model-free methodology, we follow Langer and Lemoine (2020). Following the same notation in the previous section, we re-write equation (6) as

$$\frac{O(F_{t,T_f},K,T_o)}{\bar{O}(F^L_{t,T_f},K,T_o)} = p^L_t + (1-p^L_t)\frac{\bar{O}(F^R_{t,T_f},K,T_o)}{\bar{O}(F^L_{t,T_f},K,T_o)} \triangleq \bar{p}^L \tag{9}$$

The ratio of the observed option price and the counterfactual option price in the "Leave" state (labelled as $\bar{p}^L$) is equal to the risk-neutral probability of the "Leave" outcome in addition to a bias term, which depends on the unobserved counterfactual option prices in the "Leave" and 'Remain' states. Since option prices are non-negative, $\bar{p}^L$ must be higher than p^L. Thus, $\bar{p}^L$ is the upper bound of the actual risk-neutral probability of "Leave" outcome. In order to recover $\bar{p}^L$, we reduce the bias term by choosing options with maximum price difference in "Leave" and 'Remain' states $(\bar{O}(F^L_{t,T_f},K,T_o) - \bar{O}(F^R_{t,T_f},K,T_o))$. We know that the GBP/USD spot/future rate is lower in the "Leave" state than the 'Remain' state. Thus, put option price is higher in the "Leave" state than the 'Remain' state, while call option price is the opposite. So, employing put options to recover the upper bound of the actual "Leave" probability can minimize the bias and tight the upper bound.

Assume that time $t-1$ is one day before the event day and t is the event day that outcome L is realized. Since the interval between $t-1$ and t is small enough, we can re-write equation (9) as

$$\frac{P(F_{t-1,T_f},K,T_o)}{P(F^L_{t,T_f},K,T_o)} = p^L_{t-1} + (1-p^L_{t-1})\frac{\bar{P}(F^R_{t,T_f},K,T_o)}{P(F^L_{t,T_f},K,T_o)}, \tag{10}$$

where $P(F^L_{t,T_f},K,T_o)$ is the observed put option price in the "Leave" state after the Brexit outcome release.

Empirically, we can estimate $\bar{p}^L$ in each strike price K from the following regression,

$$\ln\frac{P(F_{t-1,T_f},K,T_o)}{P(F^L_{t,T_f},K,T_o)} = \alpha_K + \beta_K Event_t + \theta_K X_t + \varepsilon_{Kt}. \tag{11}$$

where $Event_t$ is a dummy variable to indicate the information releasing day and X_t is a set of control variables, including time-to-maturity and its squared term, dummy variables to control for the days before and after the event, and dummy variables to control for a three-day window around the releasing day of the event outcome. Then we can recover $\bar{p}_K^L$ as

$$\bar{p}_K^L = e^{\hat{\beta}_K}, \tag{12}$$

where $\hat{\beta}_K$ is the estimated β_K in equation (11). Our estimated $\bar{p}_K^L$ is a set of probabilities across a dense set of strike prices. In order to fit the estimated probabilities across strike prices using a spline, we adopt the following regression

$$\ln \frac{P(F_{t-1,T_f}, K, T_o)}{P(F_{t,T_f}^L, K, T_o)} = \beta_0 Event_t + \sum_{j=1}^{J} \beta_j \min(\mu_{j-1} - K, \mu_j - \mu_{j-1}) Event_t + \theta X_t + \varepsilon_t, \tag{13}$$

where μ_j are knots that evenly divide strike prices into J groups. We set $J = 20$, which means that twenty knots are used in this regression. Then we can recover $\bar{p}^L(\mu_i)$ as

$$\bar{p}^L(\mu_i) = \exp\left(\hat{\beta}_0 + \sum_{j=1}^{20} \hat{\beta}_j \min(\mu_{j-1} - \mu_i, \mu_j - \mu_{j-1})\right). \tag{14}$$

2.4 Correcting Risk-neutral Probabilities using Stochastic Discount Factors

In addition to the primitive security prices and risk-neutral probabilities that will be obtained in the options market, we will also use data from prediction markets. The latter is much simpler to analyse than the more complex derivative asset markets. The reason is that betting odds from prediction markets provide straightforward (public) Arrow-Debrew primitive prices. The shortcoming, however, is that these prices are available (or traded) only for certain states of nature, notably those that raise public interest such as the Brexit referendum.

As suggested by macro-finance theory, there is a mapping between prices of state-contingent securities and their respective probabilities. Knowledge of Von Neumann-Morgenstern utility functions is key to unlock state or event physical probabilities that are embedded in these primitive prices. We employ two approaches to tackle this problem. The first one is based on Ross (2015)'s recovery theorem. The second assumes functional forms for investors' preferences and calibrates the model's crucial parameters.

Ross (2015) relies on a set of crucial assumptions regarding the structure of the economic environment that delivers a "non-parametric" recovery of these probabilities. The exact meaning of "non-parametric" is that the recovery is obtained without imposing any structure on the functional form of the utility function. Also, there is no need to perform any *a priori* parameterisation of the stochastic discount factor that describes the relevant economic system. In fact, the recovery does not require any knowledge of the underlying consumption, income, wealth or endowment processes that are crucial to saving decisions.

But all of this comes at a cost. Maybe the most contentious one is the assumption that both state prices and transition probability functions are time-homogenous. The theorem also considers transition independent pricing kernels, no-arbitrage and a finite state space under complete markets. A comprehensive review is provided by Carr and Yu (2012) and applications can be found in Martin and Ross (2019) and Schneider and Trojani (2019), for instance. For a criticism, see Borovička et al. (2016). We do not take a stand on the relative merit of each approach neither discuss their advantages and shortcomings in detail. However, we will point out some crucial differences between the risk correction approaches while presenting the methods and results.

2.4.1 General Recovery

The general set up of the model is given below. Consider the following Arrow-Debreu (AD) state price matrix

$$\mathbf{A_t} =: \begin{bmatrix} A_{11,t} & A_{12,t} & \dots & A_{1n,t} \\ A_{21,t} & A_{22,t} & \dots & A_{2n,t} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1,t} & A_{n2,t} & \dots & A_{nn,t} \end{bmatrix}, \tag{15}$$

where $A_{ij,t}$ is the price in state i of the AD primitive security that pays off one unit of the domestic currency if and only if state j materialises; $p(i,j)$ as the probability of occurrence of state j given the initial state i. Hence, $f(i,j)$ can be understood as a frequency or a mass function for discrete states, as in Carr and Yu (2012), for instance, or a density function for continuous states as in Ross (2015). There is a finite number of states of nature, n, in which the economy could make a transition from t to $t+1$: $i \in \{1,2\dots,n\}, \forall t$ and $t = 1,2,\dots,T$, where T is the end of the sample period.

AD primal prices Starting from state $i \in \{1, 2, \ldots, n\}$ at time t, write the following general Euler equation for a representative utility optimizer, solving it for the corresponding primitive price

$$A_{ij,t} = \beta \frac{u'(ij, t+1)}{u'(i,t)} p(ij,t), \tag{16}$$

where we consider a period utility function that is additively separable between time and states, twice continuously differentiable and follows the Inada conditions. For a risk averse agent, the function will be strictly concave; $\beta \in (0,1)$ is a subjective time-discount factor. Note that $\beta \frac{u'(ij,t+1)}{u'(i,t)}$ are pricing kernels.

The details about how to obtain the AD state price matrix from the betting and option markets is shown in Appendix 5.

2.4.2 Non-parametric Recovery

Define the inverse marginal utilities as $v(i,t) =: \frac{1}{u'(i,t)}$ and $v(ij,t) =: \frac{1}{u'(ij,t)}$. Assume that pricing kernels are transition independent, i.e. that $u'(ij, t+1) = u'(j,t)$, $\forall i$, $\forall j$ and $\forall t$ such that (16) can be simply rewritten as:

$$A_{ij,t} = \beta \frac{v(i,t)}{v(ij,t)} p(ij,t) = \frac{v(i,t)}{v(j,t)} p(ij,t). \tag{17}$$

One can stack all equations for each i, j and t in matrices, which allow us to write the system in the following compact way

$$\underset{(n\times n)}{A_t} = \beta \underset{(n\times n)}{D_t} \underset{(n\times n)}{P_t} \underset{(n\times n)}{D_t^{-1}}, \tag{18}$$

where

$$\mathbf{D_t} =: \begin{bmatrix} v_{11,t} & 0 & \ldots & 0 \\ 0 & v_{22,t} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & v_{nn,t} \end{bmatrix} \text{ and } \mathbf{P_t} =: \begin{bmatrix} p_{11,t} & p_{12,t} & \cdots & p_{1n,t} \\ p_{21,t} & p_{22,t} & \cdots & p_{2n,t} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1,t} & p_{n2,t} & \cdots & p_{nn,t} \end{bmatrix}. \tag{19}$$

Assume that all other hypothesis put forward by Ross (2015), which were made clearly explicit in Martin and Ross (2019), are valid. Entries of $\mathbf{A_t}$ are all non-negative (due to non-arbitrage) and the matrix is irreducible (due to complete markets). Perron-Frobenius theorem then ensures that $\mathbf{A_t}$ has a unique largest (in absolute value) real eigenvalue and that all elements in the corresponding eigenvector are strictly positive. Let us consider that ϕ is the eigenvalue and $\mathbf{Z_t}$ the eigenvector such that

$$\mathbf{A_t Z_t} = \phi \mathbf{Z_t}, \tag{20}$$

where

$$\mathbf{Z_t} =: \mathbf{D_t e}, \tag{21}$$

and

$$\mathbf{e} =: \begin{bmatrix} 1 & 1 & \cdots & 1(n) \end{bmatrix}^\top. \tag{22}$$

The theorem guarantees that the solution is unique up to a scalar (ϕ). The decomposition in (20) thus gives ϕ and, by Gaussian elimination, $\mathbf{Z_t}$. In Appendix B.2, we show how to obtain $\mathbf{P_t}$ with a simple example.

2.4.3 Parametric Recovery

The second approach is parametric where we need to make assumptions on the functional form of utility and risk preferences in equation 16 in order to obtain physical probability from AD prices. For example, we can define utility over wealth, and we have two representative agents with CCRA (power) utility functions (Liu et al., 2020) given by

$$u^o(W_i) = \frac{W_i^{1-\gamma^o}}{1-\gamma^o}, u^p(W_i) = \frac{W_i^{1-\gamma^p}}{1-\gamma^p}, \tag{23}$$

where o and p stand for the agent in the option and prediction market, respectively. It follows, for instance, that γ^o (γ^p) is the constant relative risk aversion coefficient for the agent in the option (prediction) market; W_i is the level of wealth in state i. Notice that we are implicitly assuming that both markets are segmented. We will later show how the range of risk aversion parameter and changes in wealth for these states of nature, that is, remain vs. leave, affect the risk correction under parametric recovery.

3 Data Description

3.1 Exchange-listed British Pound Futures and Options

We obtain daily settlement prices and intraday trade and quote prices for American-style British pound monthly futures options. We also collect the data on the underlying assets of these options, GBP futures contracts. All these options and futures contracts are listed on the Chicago Mercantile Exchange (CME). Specifically, the CME GBP option is an option written on a future contract based on 62,500 GBPs, which is quoted in USD and cents per British pound increment. The expiration date of the monthly option is the two Fridays before the third Wednesday of each month. The expiry dates of CME British pound futures are available in quarterly frequency, that is, every Monday before the third Wednesday of March, June, September, and December. The underlying future contract of an option is the nearest future contract in March, also in a quarterly frequency. The sample period used in this paper runs from January 2, 2014, to August 30, 2016. To investigate whether there is a significant difference in the British pound RNDs before and after the Brexit referendum, which was held on June 23, 2016, we mainly focus on the period between May 3, 2016, and July 29, 2016. All futures and options data is obtained from Thomson Reuters Tick History (TRTH).

3.2 Betting Odds from Prediction Markets

Many researchers use information from the prediction market to extract underlying probabilities. This is because the odds of the event reflects investors' beliefs in the probability of its outcome (Roberts, 1990; Herron, 2000; Snowberg et al., 2007, 2011; Croxson and James Reade, 2014; Meng, 2017; Auld and Linton, 2019). As the largest betting exchange with high liquidity, Betfair's information can be a valuable source of recovering the probability of the outcome of a particular event. The *odds* (payouts) of *bets* (binary contracts) are driven by the supply and demand of buy and sell contracts.

There were two contracts listed on the Betfair website for consumers to bet on the outcome of the Brexit referendum: one with a payout in the Leave state and another with a payout in the Remain state. The normalized price of the bet is $\frac{1}{odds}$, which is the cost of winning \$1 if the consumer bets in the right direction. To avoid arbitrage opportunities, the sum of the normalized prices of two contracts betting on Brexit should be close to 1. Many authors use this price as a synonym of the market-implied probability of the event outcome (Snowberg and Wolfers (2010)). In order to analyse the implied probability of Brexit in the prediction market, we collect the 5-minute odds of these two contracts from May 3, 2016 until June 24, 2016. Then we calculate the market-implied probability of "Leave" outcome as

$$p_{p,t}^{L} = \frac{1}{LeaveOdds_t}, \tag{24}$$

Figure 2a shows daily "Leave" probabilities implied in the prediction market under risk neutrality. The prediction market consistently points to "Remain" as the most likely outcome. Specifically, the daily "Leave" probabilities is around 20% to 30% between the mid of May and the end of May, 2016. It reaches the highest point on 14 June, 2016, however, it is still lower than 50%. On the night of the referendum, the 'Leave' probability implied in the betting market decreases to 16.66%.

3.3 Political Opinion Polls

Political opinion polls are an important source of information on the expected outcome of a future binary political event. The option and prediction markets reveal the probabilities of future event's outcome, while polls reflect voters' voting intentions. We collect political opinion polls data between 10 January, 2016 and 23 June, 2016. In this period, 13 market search companies (including BMG Research, ComRes, GQR, ICM, Ipsos MORI, NATCEN, ORB, Opinium, Panelbase, Populus, Survation, TNS and YouGov) published 128 polling results on whether the UK should be in or out of the EU.

Figure 2b shows that opinion polls from decided respondents favoured "Remain" majority of the time, 55% of 128 polls, though voting intentions to "Leave" that fluctuate around 50% from the opinion polls reflect the uncertainty around the Brexit outcome. In fact, several polls clustered around 14 June 2016 indicated a "Leave" outcome. In line with the polls' results, the betting market also consistently pointed to Remain as the most likely outcome (Figure 2a). However, the spike in leave probability around 14 June 2016 indicate that betting market participants take into consideration news about polling results when assessing the likelihood of Brexit outcome.

3.4 Survey Data

The British Election Study (BES) conducts a wide range of surveys for major political events in the UK, including the Brexit referendum. Respondents answer a wide range of questions designed to capture their risk preferences, demographic and socio-economic characteristics (e.g., gender, age, income), views on economy and politics, attitudes toward immigration, party identification and so on. This allows us to investigate the "Leave" voters' motivations at the micro-level, and estimate the voting intentions to "Leave" of *indecisive* voters.

We use the individual level data from waves 8 of the 2014–2017 BES Internet panel. This survey is conducted between 6/May/2016 and 22/June/2016 and covers 33,501 individuals. For the question *"If you do vote in the referendum on Britain's membership of the European Union, how do you think you will vote?"*, 15,215 respondents (45.42%) answer *"Stay in the EU"*, 15,793 respondents (47.14%) answer *"Leave the EU"*, 475 respondents (1.42%) answer *"I would not vote"* and 2,018 respondents (6.02%) answer *"Don't know"*. The latter group is of a particular interest since a large fraction of such voters are likely to determine the final Brexit outcome.

In order to investigate the "Leave" voters' motivations, we select a series of explanatory variables, including risk preferences, personal annual gross income as well as the variables used in earlier studies (e.g., Goodwin and Milazzo (2017)). The details of these variables and corresponding BES questions are listed in Table 1. To keep sufficient number of observations, we use the median value of the explanatory variable to replace the "Don't Know" answers.[4]

3.5 Additional Data

In addition to the data from financial market, polls and survey, we also collect data for RND extraction and market fear measurement. In particular, we collect GBP/USD spot exchange rate between January 2, 2014 to August 30, 2016 from Bloomberg. For risk free rates, we use US and UK LIBOR rates obtained from Federal Reserve Bank of St. Louis website. In order to match

[4] Our main results are not sensitive to this assumption, results of alternative specifications are available upon request.

futures and options maturities, we interpolate interest rates for specific horizons. Then we use the interpolated interest rate for further analysis.

To study the trading behavior of different type of traders in the CME British pounds derivative market, we obtain the publicly data from the Commodity Futures Trading Commission (CFTC). The weekly Commitment of Trader (COT) Reports contains the aggregate long and short positions for three types of traders: commercials, non-commercials and non-reportable, and spread positions for non-commercials. Following the literature, we view commercials as hedgers, non-commercials as speculators and non-reportable as small speculators.

4 Empirical Results

4.1 Implied Volatility and Risk-Neutral Density

To extract information about investors' beliefs on the performance of GBP/USD exchange rates and investors' attitudes towards risk before and after the Brexit referendum, we extract RNDs of GBP/USD futures from CME British pound monthly futures options.

Figure 3 display implied volatilities and RNDs inferred from CME British pound monthly futures options expiring on 8 July, 2016, which is the earliest expiry day after the referendum date. Specifically, Figure 3a and Figure 3c show implied volatility surface and its corresponding RND surface during June 2016. We select six specific days to plot their implied volatility curves and their corresponding RNDs in Figure 3b and Figure 3d, including 6-, 4-, 2-weeks before the referendum, the referendum day and 1-day and 1-week after the referendum.

Implied volatility surface of options expiring on 8 July, 2016 in Figure 3a shows the implied volatility change across a wide range of strike prices during June 2016. For a specific day before/at the referendum, the implied volatility decreases as the strike price increases, that is, the implied volatility curve of British Pound options exhibits a well-known 'smirk' shape. Fig. 3b shows that implied volatility curves in the shape of 'smirk' can be observed in the market at least from 12 May, 2016. It indicates that OTM puts are more expensive than ATM options and OTM calls. This option price differences across strike prices are driven by demand differences. That means, to avoid the possibility of a crash or tail risk, investors in British Pound option market have high demand of OTM puts before the Brexit outcome releases, and this demand become even higher as the outcome releasing day approaches. After the outcome released, the implied volatility curve gradually returns to the normal 'smile' shape.

Based on the fitted implied volatilities, we approximate the RND of GBP/USD futures and append it into left and right tails. The RND surface in Figure 3c displays the RND movement from the beginning of June until the end of June in 2016. As we have seen, when the time approaches the referendum day, RND shows a more pronounced bimodal distribution, with a major mode towards the high price region and a minor mode towards the low price region. This suggests that a sharp rise or fall should be expected relative to the current level of GBP/USD futures once the Brexit outcome is realized. Taking the RND on the referendum day as an example, the red line in Figure 3d, we find that the major mode is 1.525 and the minor mode is 1.335, which represent the most likely GBP/USD futures prices if the Brexit outcome is 'Remain' or "Leave", respectively. After the Brexit outcome is realized, the market reaches a consensus and RND of GBP/USD futures go back to normal unimodal distributions.

After investigating the time-variation of RNDs during May and June in 2016 by using options expiring on 8 July, 2016, we examine the term structure of RNDs before and after the Brexit outcome is realized. In Figure 4, we display RNDs for all available horizons on the referendum day (Figure 4a) and the first day that after the result of the Brexit referendum (Figure 4b). On 23 June, 2016, the RND extracted from options expiring on 8 July, 2016 shows the strongest bimodal distribution. The RNDs extracted from options expiring in the following three months are slightly bimodally distributed. While RNDs for long horizons are more likely unimodally distributed. This implies that investor's expectations about short-term changes in the GBP/USD exchange rate are more divergent than their long-term forecasts. Similar to the pattern unveiled for changes in the RND that we observed from the option that expires on 8 July, 2016, RNDs from options that expire later also return to a normal unimodal distributions on the 24 June, 2016, as the result of the referendum is certain.

4.2 The Dynamic Behavior of the British Pound's Risk-Neutral Distribution

To study the dynamic behavior of the British Pound's RND, we extract risk-neutral moments from the RND of British Pound futures prices. Movements of risk-neutral moments can provide information about the daily changes in investors' expectations of the futures underlying prices and its associated uncertainty. Specifically, we focus on the risk-neutral second-, third- and fourth-moments, and the bi-modality coefficient measured by risk-neutral skewness and kurtosis. The time t bimodality coefficient (BC) of a RND from a contract expiring at T_o is

$$BC = \frac{Skewness^2 + 1}{ExcessKurtosis + \frac{3(n-1)^2}{(n-2)(n-3)}}, \tag{25}$$

where n is the sample size. The range of BC is from 0 to 1. $BC > \frac{5}{9}$ indicates that the distribution maybe bimodal or multimodal.

Figure 5 presents the annualized risk-neutral volatility, the risk-neutral skewness, the risk-neutral excess kurtosis and the bimodality coefficient from RNDs for five different horizons between 2 May, 2016 and 29 July, 2016. Since the prices of the GBP/USD futures expiring before the referendum day are not affected by the "Brexit" outcome, the risk-neutral moments of RNDs from an option that expires before the referendum are at normal levels, and RNDs have normal unimodal distribution. To further analyze the behavior of RNDs extracted from options expiring after the referendum, we use information in RNDs obtained from this option as a reference for normal levels.

The risk-neutral volatility (Figure 5a) represents the market's uncertainty around the expected value of GBP/USD futures. Risk-neutral volatilities obtained from options expiring within three-month after the referendum are higher than a normal level. Starting in early June 2016, risk-neutral volatilities from options expiring after the referendum increase significantly until the Brexit outcome is realized, especially for volatilities from short maturity options. This means that the uncertainty of Brexit has a larger impact on the short-term value of the GBP/USD exchange rate than its long-term value. After the announcement on June 24, 2016, the risk-neutral volatility from options expiring in July and August drop back to their level in May within the next few days, but the risk-neutral volatility from longer maturity options take more time to recover to May levels. The change in risk-neutral volatility indicates the resolution of the uncertainty caused by the referendum. We can also use this change to infer how much information is provided by the announcement of the result. According to the empirical evidence presented above, agents use this information to settle short-term uncertainty, rather than long-term uncertainty. This finding is not surprising, since the procedures on how to leave the EU and the potential economic agreements between the EU and the UK would still be uncertain.

The risk-neutral skewness (Figure 5b) is a measure of asymmetry. Before the Brexit referendum, the risk-neutral skewness from options expiring after the referendum is more negative than the normal level, which indicates that RNDs have fat left tails. Specifically, risk-neutral skewness from options with short maturities is more negative than skewness from long maturities options. This is induced by the risk of "Leave". As the "Leave" result would represent a negative shock to GBP/USD exchange rate, especially in the short-horizon, the risk of "Leave" is shown as fat left tails in RNDs. As the day of the Brexit referendum approaches, risk-neutral skewness become even more negative, reflecting the market expectations of a negative shock (Hasler and Jeanneret, 2020). These unusual fat left tails that appeared a few days before the referendum are more likely to be prompted by bi-modally distributed RNDs. As the state of the world "Leave" is realized and the risk-neutral skewness goes back to a normal level, which is close to zero, RNDs return to normal symmetric distributions.

The risk-neutral excess kurtosis (Figure 5c) measures market expectations of extreme changes in the GBP/USD exchange rate. The higher the excess kurtosis, the higher the probability concentrated in the tails of RNDs before the Brexit referendum. Similar to the third moment, the excess kurtosis also goes back to a normal level after the referendum.

Last, we examine the bi-modality coefficient (Figure 5d) calculated from the risk-neutral skewness and excess kurtosis. The time-varying BC indicates that bimodal RNDs from short maturity options can be observed within three days before the referendum, while RNDs from December contracts are consistently uni-modally distributed. After the referendum, the BC falls back to a normal level, which means that RNDs also return to normally unimodal distributions. Hence, regime switches of the moments of RNDs and changes of their shapes after the referendum are likely reflecting the fact that the outcome was not entirely unexpected.

4.3 The Tail Risk

To investigate the tail risk in the exchange rate, we calculate risk-neutral 10% quantiles of the GBP/USD spot rate return according to the definition in equation (1).

Figure 6a shows the term structure of tail risk of the GBP/USD exchange rate from January 2014 to August 2016. Apart from Brexit, this sample period covers two additional important events in the UK, namely the Scottish independence referendum on 18^{th} of September, 2014 and the UK general election on the 7^{th} of May, 2015. Since the available maturity for the options is up to one year in our sample and the trading in long maturity contracts is relatively thin, we calculate risk-neutral 10% quantiles for 1-month, 3-month and 6-month horizons. The stark declines in 10% quantiles before the event indicate that the uncertainty of the event outcome has triggered an increase in tail risk. Compared with the other two events, the tail risk caused by Brexit is much larger across all three horizons. In addition, for both the Scottish independence referendum and the United Kingdom general election, we can observe 10% quantiles steeply rises to the level of the no-event period on the day of the announcement of the results. The Brexit referendum is different: the tail risk returns to normal levels more than 1-month after the binary event. This evidence suggests that the impact of the Brexit referendum on the level of tail risk is significantly different from the Scottish independence referendum and the 2015 United Kingdom's general election, at least for the GBP/USD foreign exchange market.

In Figure 6b provides a closer look at the period around the Brexit referendum, and shows that the tail risk deviates from its normal level since the beginning of 2016, especially after April 2016. It reflects the investors' forward-looking assessment of a higher likelihood of the "Leave" outcome. Using the sample period between 3 May, 2016 and 23 June, 2016, the correlation between the "Leave" probability implied in Betfair and the tail risk measure for 1-month, 3-month and 6-month horizons are -0.5123, -0.6024 and -0.6092, respectively. This result lends support to the view that the more likely the adverse state is, i.e. the "Leave" state, the higher is the degree of tail risk. Moreover, the tail risk is highest on 14 June, 2016, on the day when several poll results suggesting "Leave" outcome were released.

4.4 Option-Implied Event Probability: Model-Based Estimation

Information extracted from RNDs suggests that the CME British Pound option market contains investors' expectations for the Brexit result. To infer the investors' expected probability of the "Leave" outcome in the option market, we used the model-based methodology presented in section 2.3. This approach also allowed us to identify the implied futures prices and volatilities in "Leave" and 'Remain' states, respectively.

Figure 7 presents the daily option-implied probability of "Leave" using three different model specifications, together with the corresponding implied probability in the betting market. The option-implied probabilities from different model specifications are highly correlated (correlation greater than 0.85), but have different magnitudes, especially before 14 June, 2016. Compared with our baseline model, model specification II and III attribute a large weight on cheap options, since cheap OTM options likely to carry important information about speculation related to tail events. These specifications reduce the level of estimated option-implied “Leave” probabilities (closer to risk-neutral probabilities from betting market), especially before the 14 June 2016. Apart from that, we find that daily “Leave” probabilities in the option market and the betting market mainly move together during this period. The correlation between probabilities from the betting market and option-implied probabilities under the three different specifications are 0.44, 0.42, and 0.47, respectively. In the betting market, the “Leave” probability reaches its highest on 14 June, 2016, which is consistent with the “Leave” probability from the poll of opinion polls (Wu et al., 2021). However, under model specification I and II, “Leave” probability from the option market reaches its highest on 16 June 2016. If we switch to the model specification III, the “Leave” probabilities on 14 and 16 June, 2016 are almost in the same magnitude. This suggests that the reaction of the option market to the murder of the pro-Remain MP Jo Cox on 16 June, 2016 is stronger than the betting market. Overall, the average probability of a “Leave” outcome provided by the options market (35.98%, 32.89% and 29.25% under the three model specifications, respectively) is higher than that of the betting market (27.43%).

We also plot the futures prices and their corresponding volatilities in the “Remain” and “Leave” states estimated under the different model specifications in Figure 8. The futures prices in these two states seem to move together in May 2016, and then start to diverge. This maybe indicate that investors in the CME British pound option market starts to pay more attention to Brexit news in June. In addition, the futures prices in the ‘Leave‘ state reach the lowest value on 13 June, 2016 under model specifications II and III, and on 14 June, 2016 under model specification I. This pattern matches the pattern of tail risk. Tail risk for 1-month horizon reaches its peak on 13 June, 2016 and for 3- and 6-month horizons peaked on 14 June, 2016. Moreover, the futures volatilities in two states show that the volatilities in the “Remain” state are much lower than ”Remain” state.

4.5 Option-Implied Probability: Model-Free Method

Option-implied probability of the Brexit outcome varies across different model specifications. To evaluate the time-varying “Leave” probability estimated from the model-based method, we extract a upper bound for the “Leave” probability by using the model-free method demonstrated in Section 2.3.2. For this regression, we use all available put option data between 2 May, 2016 and 29 July,

2016. We indicate event date as 24 June, 2016, since it is the date that the Brexit outcome was released.

Figure 9 shows estimated "Leave" probability upper bounds. The red circles represent the estimated probability by using strike-price-by-strike-price regressions in equation (11). The black line is the fitted spline with 20 knots by using the regression specified in equation (13). The vertical black dashed line represents the GBP/USD exchange rate on the day of the event, so the left side of the line are strike prices of OTM put options. The minimum probability on the fitted spline line is 33.69%, which is our preferred estimate of the "Leave" probability's upper bound. Ideally, the tightest bound should occur on a strike price of an OTM option. However, OTM put options usually carry a crash risk premium. Even if the realized Brexit outcome is "Leave", OTM put option prices may retain most of its value. Then the bias term in equation (11) does not converge to zero. The probability estimated from OTM put options provide a relatively loose bound. In our results, the minimum probability, 33.69%, occurs on the right side of the vertical line. This means that 33.69% from in-the-money (ITM) put options is not the tightest bound for the "Leave" probability. But it is sufficient for us to rule out the model specification I from our model-based estimation, because the average probability estimated from this specification is higher than the upper bound. The average probability from model specification II is very close to the upper bound, but our estimated upper bound is a relatively loose upper bound, so we believe that the estimation of the model specification III is likely to be more accurate.

4.6 Learning from the Opinion Polls

We use the "Leave" voting intention from political opinion polls to extract a proxy of *subjective* "Leave" probability. First, we convert the voting intention into a binary Brexit outcome. If the voting intention to "Leave" is higher than "Remain" from the political opinion poll i at day t, we consider the outcome as "Leave", and vice versa. We use value 1 to indicate "Leave" outcome, and 0 to indicate "Remain" outcome, which is labelled as $\mathbb{1}_{i,t}^{L}$. Second, we assume that different polls independent observations, and if there are several polls on a day, we assign the average of all available polls' outcomes at day t as the outcome at day t, $\frac{1}{K}\sum_{i=1}^{K}\mathbb{1}_{i,t}^{L}$, where K is the number of available polls at day t. However, the probability of the "Leave" outcome from individuals perspective is not only based on the polls' outcome on a single day, the past polls results also contribute to individuals' belief formation on the "Leave" outcome. Thus, in the third step, we allow for a learning mechanism from the past N days polls' results, and the weight for day-j information is $\omega_{j,t,N}$, where $t-N+1 \leq j \leq t$, then the *subjective* probability of "Leave" outcome from polls is

$$p^L_{poll,t} = \frac{\sum_{j=t-N+1}^{t} \omega_{j,t,N} \frac{1}{K} \sum_{i=1}^{K} \mathbb{1}^L_{i,j}}{\sum, m^t_{j=t-N} \omega_{j,N}}, \tag{26}$$

We consider 18 (2x3x3) different learning mechanisms, varying the estimation window, weights for past observations and number of past days. First, we consider both expanding (N is increasing) and rolling window (N is constant) estimation. Second, we use three different weight functions, since individuals may process past information differently. The first function is the equal-weighted average of past polls outcomes $\omega_{j,t,N} = \frac{1}{N}$, which assumes individuals gives equal importance to all past opinion polls. The second function is the liner decay weight function $\omega_{j,t,N} = \frac{N-(j-t)}{N}$, which assume that individuals treat the past information less important than the new information. The last weight function is the exponential decay weight function $\omega_{j,t,N} = (\frac{N-(j-t)}{N})^3$, which gives even lower weight to the past information compared to the linear decay function. Finally, we use three different window lengths, 30, 60 and 90 days, respectively. For the expending window method, window lengths means the number of days that we start the probability calculation before the beginning day of our sample.

Table 3 shows the correlations between the daily change of subjected probabilities estimated from past opinion polls and the daily change of risk-neutral probabilities from option and prediction markets. We note that the subjective probabilities using the linear decay function with a 90-day rolling window best describes the learning mechanism of both option and prediction market participants, which we exhibit in Figure 10. The correlations between daily changes in Leave probabilities from polls and option (prediction) markets, is 0.3 (0.43). Using 18 different methods to calculate subjective probability, the average correlations between the subjective and option (prediction) market-implied risk-neutral probabilities is 0.21 (0.33). This indicates that the option and prediction markets incorporate the information from polls to a large extent.

4.7 Correcting the Risk-neutral Probability

Because the referendum is a binary event, we assume that the Arrow-Debreu (AD) state price matrix, $\mathbf{A_t}$ shown in equation (15), is 2×2. State two represents the UK leaving the EU in the referendum (leave), whereas the current (remain) state is one. We make two other assumptions to complete the second row of $\mathbf{A_t}$ in the non-parametric recovery. They are not necessary for the parametric risk-adjustment. The first assumption is of “high uncertainty” state 2 as embedded in the following prices: $A_{21} = A_{22} = 0.5$. The second corresponds to an (almost) absorbing state 2: $A_{21} = 0.01$ and $A_{22} = 0.99$.

For the parametric recovery, we experiment with several plausible values of the deep parameters assuming standard power utility. Start with $\beta = 0.99$, which is a standard value in the RBC/DSGE literature for annual data and that is also consistent with our previous findings using the recovery theorem. We need to make an assumption about relative wealth change between the remain and leave states. According to a poll conducted by Ipsos Mori before referendum in 2016 (The Guardian, 2016), 25% of the voters except a decline in their living standards due to Brexit, while 14% of voters believe in an improvement in living standards, with 51% expecting no change. Hence, in our baseline setting, we assume that there will be a 10% decline in aggregate wealth if we move from “Remain” to “Leave” state and also analyze the impact of more drastic wealth changes. In our parametric risk correction, we consider both risk-averse and risk loving preferences with a constant relative risk aversion γ ranging from -6 and to 6.

Non-parametric recovery. Figure 11 presents the results using the non-parametric approach, which are shown in Figure 11a for the prediction and in Figure 11b for the option markets. While the non-parametric recovery is fairly general without imposing much structure, it generates only a very small upward correction of the leave probabilities. Notice that the ”uncertain” case presents a smaller difference in the corrected probabilities than the ”absorbing” case, however, the differences are minor. Thus, we next move to the parametric correction to check whether the level of leave probability could be larger under additional parametric assumptions to pin down the stochastic discount factor (SDF).

Parametric correction. Figure 12 shows risk-corrected leave probabilities from both markets under different parametric assumptions for risk preferences and relative wealth changes from remain to leave state. In the upper panels, we assume the wealth in “Leave” state is 10% (W^L) lower than the “Remain” state (W^R), which means $\frac{W^L}{W_R} = 0.90$, we calibrate risk-corrected probabilities with a set of relative risk aversion coefficients, $\gamma = \{-6, -4, -2, 0, 2, 4, 6\}$, where positive γ represents risk averse and negative γ represents risk loving behaviour. Results for prediction and option markets are displayed in Figure 12a and 12b, respectively. As can be seen, the assumption of risk averse agents decreases the probabilities of the leave state. A possible explanation is that primitive prices already reflect higher relative demand for that worse - by assumption - state of nature. On the other hand, the assumption of risk loving agents increase leave probabilities.

Speculative Markets? Is it plausible to assume that market participants on aggregate are risk-seeking? While it is conceivable that the prediction markets are populated by risk-seeking agents, derivative markets are often used both for hedging (by risk-averse agents) and speculative (by risk-seeking agents) purposes, especially around rare events (Bond and Dow, 2021). Whether one trading motive dominates the other remains to be an empirical question.

First, we note in Figure 13a, the trading activity in the option market proxied by the put-call ratio, is closely linked to the uncertainty of opinion poll results about Brexit outcome. The latter is measured by the standard deviation of the expected "Leave" outcome under the assumption of binomial distribution. Both the uncertainty and the put-call ratio jump on 9 June, 2016. Clearly, relative high demand for put contracts on GBP could originate both from bearish speculative bets on the British pound and/or hedging needs to protect against a large drop in GBP (Hanke et al., 2018).

Figure 13b shows speculative versus hedging behaviour in the CME British pounds derivatives markets around the Brexit event. We measure the speculative versus hedging behaviour as total CME British pounds futures' and options' positions held by speculators (non-commercial traders) relative to the total positions held by hedgers (commercial traders). To eliminate potential maturity effects, we scaled it by the values in the previous year. We notice that relative speculative activity is abnormally high around mid-June (7 June and 14 June) which coincides with poll results suggesting a leave outcome together with heightened uncertainty, and higher leave probabilities estimated both from prediction and option markets. The level of relative speculative position is lower in the last week before the referendum after the murder of the pro-remain MP Jo Cox and Bank of England's warning about the potential catastrophic implications of Brexit outcome.

Relative Wealth. We next assume mild risk seeking behaviour in both prediction and option markets ($\gamma = -2$), we obtain the risk-adjusted probabilities by using a wide range of relative wealth in "Remain" and "Leave" states, $\frac{W^L}{W_R} = \{0.75, 0.90, 0.95, 1.00,\ 1.05, 1.10, 1.25\}$. The figures 12c and 12d show the results for prediction and option markets, respectively. We see that only a large wealth decline (about 25%) from "Remain" to "Leave" state brings the risk-corrected leave probabilities closer to the physical probabilities estimated from the BES survey.

4.7.1 Risk-Corrected versus Subjective Probabilities

Using the subjective probabilities inferred from past opinion polls as the benchmark, we search for the corresponding relative risk reversion coefficient that minimizes the distance between risk-corrected probabilities and subjective probabilities. Assume $\beta = 0.99$ and the relative wealth $\frac{W^L}{W^R} = 0.90$, we can calibrate relative risk reversion coefficient, γ^{i*}, as

$$\gamma^{i*} \equiv \arg\min_{\gamma} \sum_{t=1}^{T} (\tilde{\pi}_t^i - \hat{\pi}_t)^2, \tag{27}$$

where $i = \{p, o\}$, p and o represent prediction and option markets, respectively. T is the sample size, $\tilde{\pi}_t^i$ is the risk-corrected probability by using by date from option or prediction markets, and $\hat{\pi}_t$ is the subjective physical probabilities inferred from polls.

Figure 14 shows the risk-corrected probabilities by using data from both prediction and betting markets. The resulting risk aversion coefficient equals -1.83 in the prediction market ($\gamma^{p*} = -1.83$), and equals -1.01 in the option market ($\gamma^{o*} = -1.01$). Hence, by assuming that wealth in the ”Leave” state would be 10% lower than the ”Remain” state (baseline setting) and that risk-corrected probabilities are the closest to agent’s perception of the physical probabilities, we find that the average agent in both prediction and option markets are mildly risk seeking, with stronger risk seeking preferences in prediction markets as one would expect.

Then, we relax the assumption about the relative wealth in the ”Leave” and ”Remain” state, which was $\frac{W^L}{W^R} = 0.90$ and allow the relative wealth to be in a wider range, between 0.5 and 1.5, which means the wealth in the ”Leave” state could be maximum 50% lower or higher than the ”Remain” state. Then we calibrate the corresponding relative risk reversion coefficients by minimizing the squared difference between risk-corrected probabilities and subjective probabilities.

Figure 15 shows the corresponding relative risk reversion coefficients in both prediction and option markets with respect to the relative wealth. If the agents on average expect lower wealth in the ”Leave” state than the ”Remain” state ($\frac{W^L}{W^R} < 1$), the average agent in both prediction and option markets are risk loving as in the baseline case. Expectation of a mild decrease in wealth in the ”Leave” state can only be rationalized by extreme risk-seeking behavior. If we assume that the agents on average expect a higher wealth in the ”Leave” state than the ”Remain” state ($\frac{W^L}{W^R} > 1$), then average agent in both markets is risk averse. However, symmetrically, the average risk aversion is low only if the agents on average expect a large increase in wealth moving to the "Leave" state which is empirically less plausible given the survey results before the Brexit referendum and Bank of England warnings.[5] The average agent in prediction markets has always a stronger risk preference than the average agent in option markets.

4.8 What Do We Learn from the Individual-level Survey Data?

To obtain the voting intention to "Leave" from individual-level BES survey data, we conduct Probit models to understand "Leave" voters underlying motivations to change the status quo. The Models (1) specification is using risk preferences as the only explanatory variable. Then we add voters' demographic characteristics, like gender and income, in Model (2). After that, we add voters' political views as additional variables in Model (3). Furthermore, we add voters immigration attitudes in Model (4). Last, we use all potential explanatory variables in the regression as Model (5).

Table 2 shows that *Risk Taking Preference* is one of the key motivation of respondents' "Leave" votes. Specifically, voters with higher risk taking preferences would like to vote for "Leave", which is consistent across different model specifications suggesting that controlling for other factors, risk seeking attitude is one of the important drivers of "Leave" votes. The most comprehensive Model (5) shows that being Male, having lower income, having left school early, belonging to elderly group also increase the likelihood of "Leave" vote. Voters with more concerns about immigration and more pessimism about economy prefer to vote "Leave". Party Identification also plays an important role for voters' choice.

In the bottom part of this table, we show both the share of leave voters and estimated intention to "Leave". The vote intention of "Leave" based on "Leave" and "Remain" respondents is 50.93%. By employing the estimated coefficients and indecisive respondents' individual characteristics, we obtain "Leave" probably implied for indecisive voters, which is higher than the probability of "Leave" vote implied for "Leave" and "Remain" voters under specification (4) and (5). According to model (5), the total intention to "Leave" is 51.07% including indecisive respondents. Hence, taking into account the indecisive voters (about 6% of the sample) suggests even a higher likelihood of Brexit outcome.

[5] One could argue that the Brexiteers were very optimistic about their wealth improvement post-Brexit, however, the BES survey results suggest that leave voters are more risk-seeking compared to remain voters.

We also provide daily estimates of intention to leave using the most comprehensive model (5). Figure 16 shows that the intention to "Leave" is more persistent than the polling results. Arguably, one could extract a better proxy of physical Brexit probability based on survey estimates of voting intentions. However, these survey results were not available to market participants before the Brexit referendum. Ex-post we learn that individual characteristics that are persistent over time are a better leading indicator for the Brexit outcome.

5 Concluding Remarks

We find that risk-neutral density distributions extracted from British Pound options with short maturities are bimodally distributed one week before the Brexit referendum. The changes of the risk-neutral tail risk in GBPs are highly correlated with changes of "Leave" probabilities from the betting market. This empirical evidence shows that the GBP options market incorporates investors' beliefs about a potential Brexit outcome. We estimate the risk-neutral probabilities of the Brexit referendum from the options market by using both model-based and model-free methods. The risk-neutral probability from the option market on average is higher than the probability from the betting market in the month before the Brexit referendum, but both market participants seem to closely track opinion poll results when assessing event probabilities. While subjective probabilities extracted from polls rationalize the Brexit surprise, voting intentions estimated from surveys which are determined by persistent characteristics (age, education, income), political views and the risk preferences of the voters, are likely to be a better guide for physical probabilities.

We construct risk-neutral Arrow-Debrew prices from both markets and filter out risk-corrected probabilities from market prices using both a non-parametric (Ross Recovery Theorem) and a parametric (calibrating the stochastic discount factor) approach. Only parametric recovery is likely to have an impact on the level of Brexit probability estimates, albeit under strict parametric assumption. However, we argue that markets could have signalled higher Brexit outcome once we allow for speculative trading triggered by such binary political events in both prediction and option markets. Arguably, reliance on risk-neutral probabilities from both markets were misleading as an indicator for "Leave" outcome.

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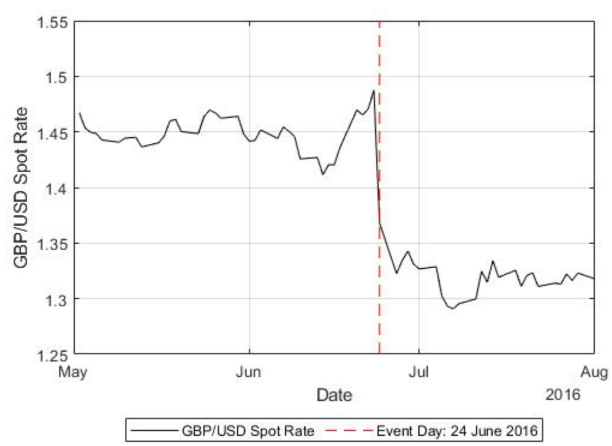
Figure 1: GBP/USD Exchange Rate and Put Option Price

This figure presents the GBP/USD spot exchange rate and the CME British Pound put options price change around the Brexit referendum. Figure 1a displays the GBP/USD spot exchange rate between May and July, 2016. Figure 1b displays price changes of the CME British Pound put options on the event day and also non-event days.

(a) GBP/USD Exchange Rate



(b) Put Option Price Changes

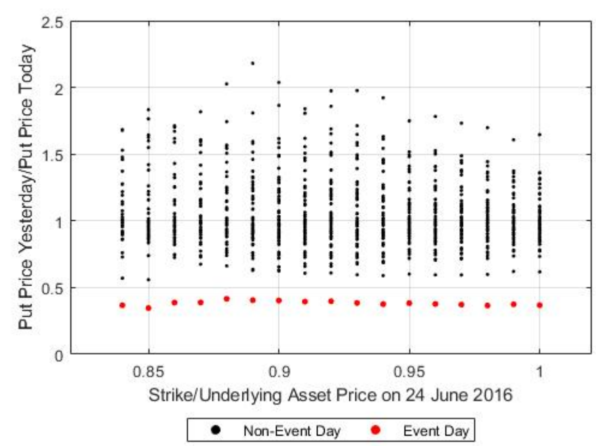
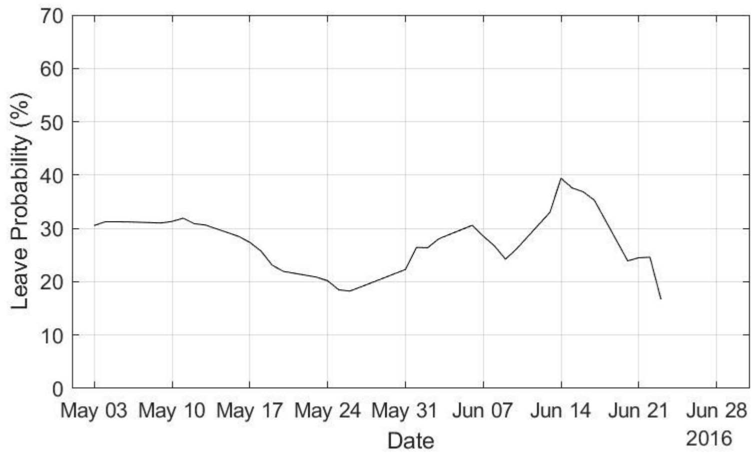


Figure 2: Prediction Markets versus Political Opinion Polls

This figure presents the implied probability of "Leave" obtained from prediction market and the percentage of Leave voters in several polls. Figure 2a shows the "Leave" probability recovered from predication market by using Betfair "Leave" odds between 3 May, 2016 and the 23 June, 2016. Figure 2b shows the voting intentions of "Leave' from thirteen market research companies (see legend below for dots, triangles and squares, n=128) between 10 January, 2016 and the 23 June, 2016. The voting intentions of "Leave' are measured as "Leave" percentage of decided respondents from political opinion polls.

(a) Leave Probabilities from Prediction Market



(b) Voting Intention from Political Opinion Polls

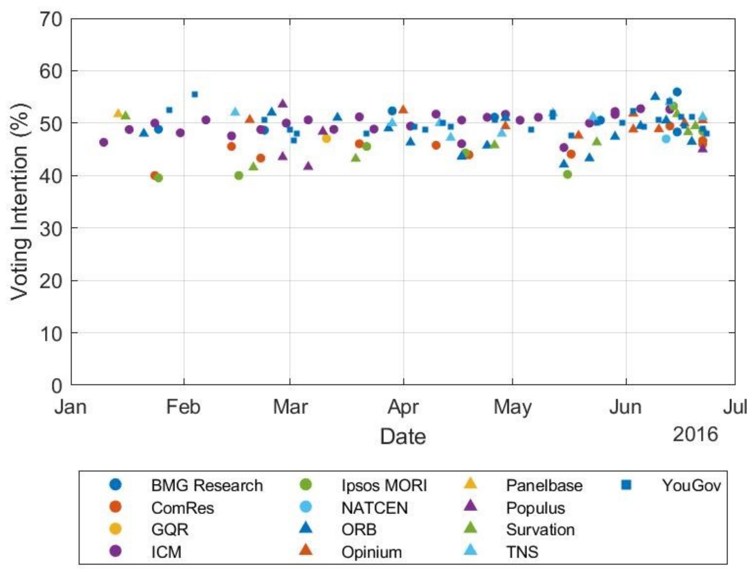
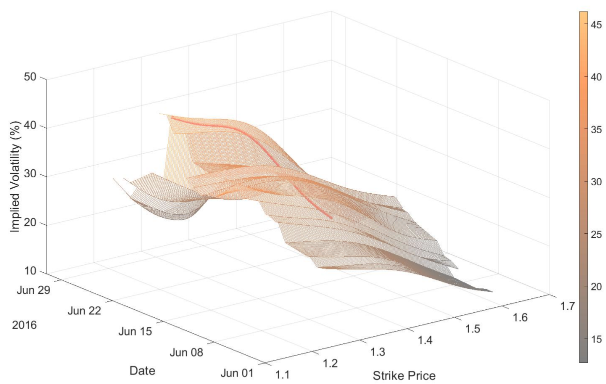


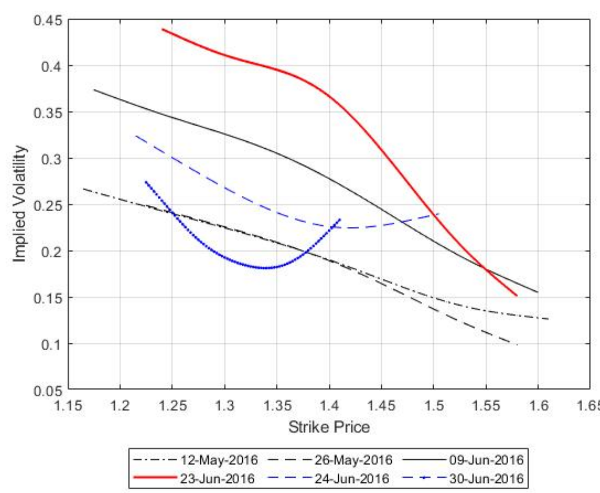
Figure 3: IVs and RNDs Inferred from British Pound Options expiring on 8 July, 2016

This figure presents fitted implied volatilities (Figure 3a and Figure 3b) and risk-neutral density distributions of GBP/USD futures (Figure 3c and Figure 3d) inferred from CME British Pound monthly futures options expiring on 8 July, 2016. Figure 3a and Figure 3c display the fitted implied volatility surface and RND surface of GBP/USD futures between 1 June, 2016 and 30 June 2016 across a wide range of strike prices. The red lines in these two figures indicates the Brexit referendum day (23 June, 2016). Figure 3b and Figure 3d are the fitted implied volatility curves and the corresponding RNDs of the six specific days, 6-, 4-, 2-weeks before the Brexit referendum (12 May, 2016, 26 May ,2016 and 09 June, 2016), the Brexit referendum day (23 June, 2016) and 1-day and 1-week after the Brexit referendum (24 June, 2016 and 30 June, 2016). In these two figures, black, red and blue lines represent the dates before, at and after the Brexit referendum, respectively.

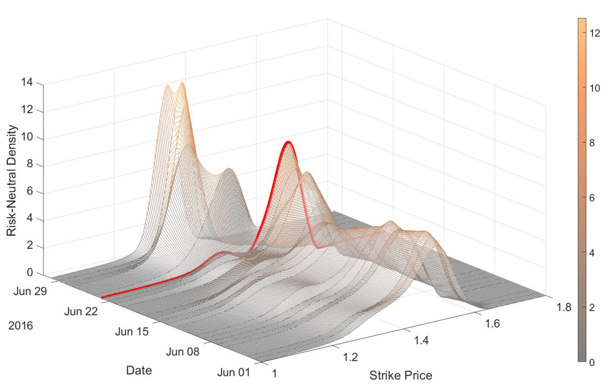
(a) Implied Volatility Surface



(b) Implied Volatility in Different Days



(c) Risk-Neutral Density Surface



(d) Risk-Neutral Densities in Different Days

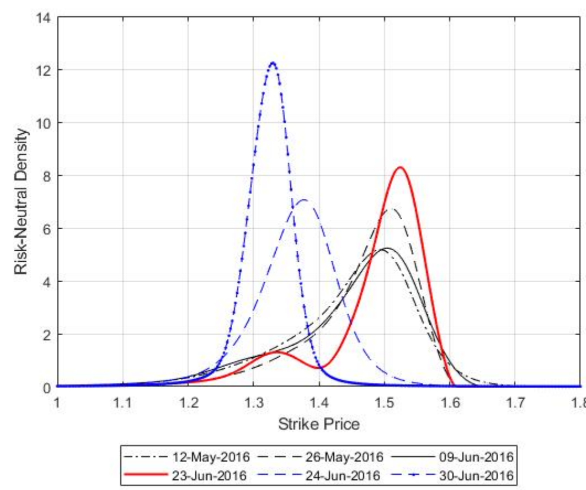
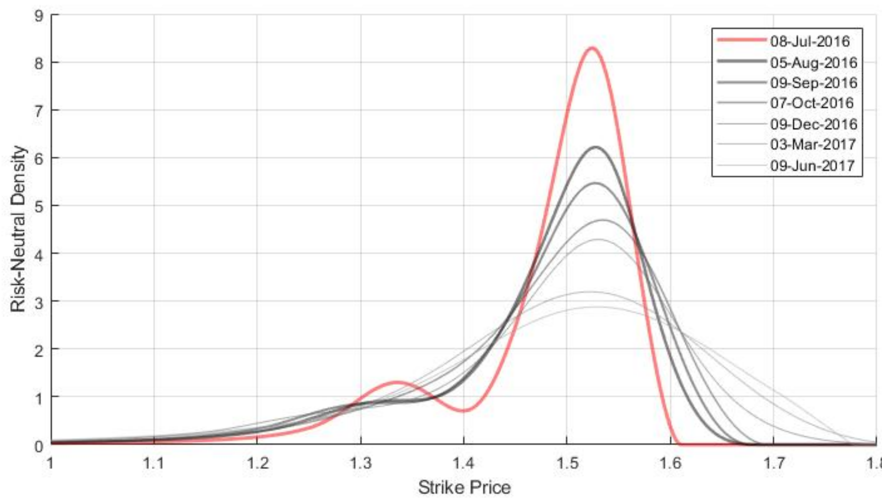


Figure 4: RNDs Inferred from British Pound Options on 23 and 24 June, 2016

This figure presents risk-neutral density distributions of GBP/USD futures inferred from British Pound options across all available maturities on 23 June, 2016 (Figure 4a) and 24 June, 2016 (Figure 4b). Red lines are RNDs extracted from options expiring on 8 July, 2016. Black/gray lines are RNDs extracted from options expiring after 8 July, 2016, between 5 August, 2016 and 9 July, 2017.

(a) Risk-Neutral Densities on 23 June, 2016



(b) Risk-Neutral Densities on 24 June, 2016

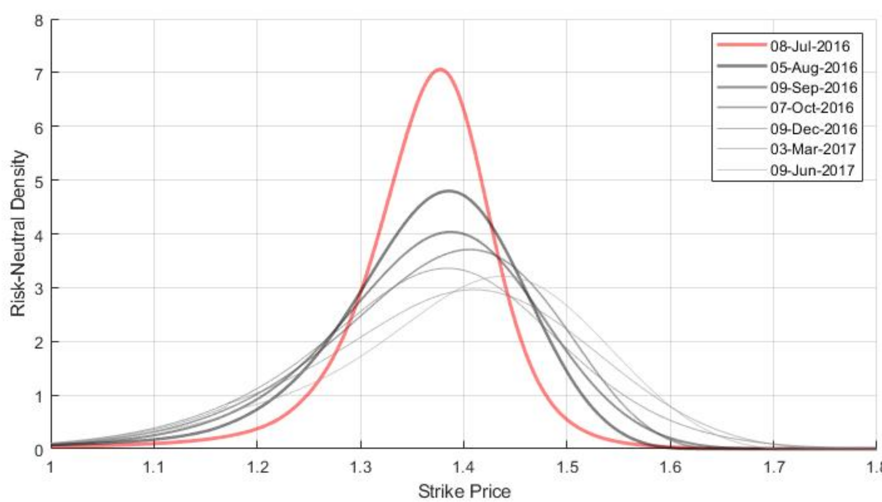


Figure 5: Information in RNDs Inferred from British Pound Options

This figure presents presents the annualized risk-neutral volatility (Figure 5a), the risk-neutral skewness (Figure 5b), the risk-neutral excess kurtosis (Figure 5c) and the Bimodality Coefficient (Figure 5d) of British Pound futures prices between 2 May, 2016 and 29 July, 2016. In this figure, we focus on RNDs from options with five different maturities, one maturity is before the Brexit referendum (3 June, 2016, blue line), three within the next following three months after the Brexit referendum (8 July, 2016 (red line), 5 August, 2016 (black line) and 9 September, 2016 (dark gray line)) and one six-month after the Brexit referendum (9 December, 2016, light gray line). The vertical line shows the day of the Brexit referendum on 23 June, 2016.

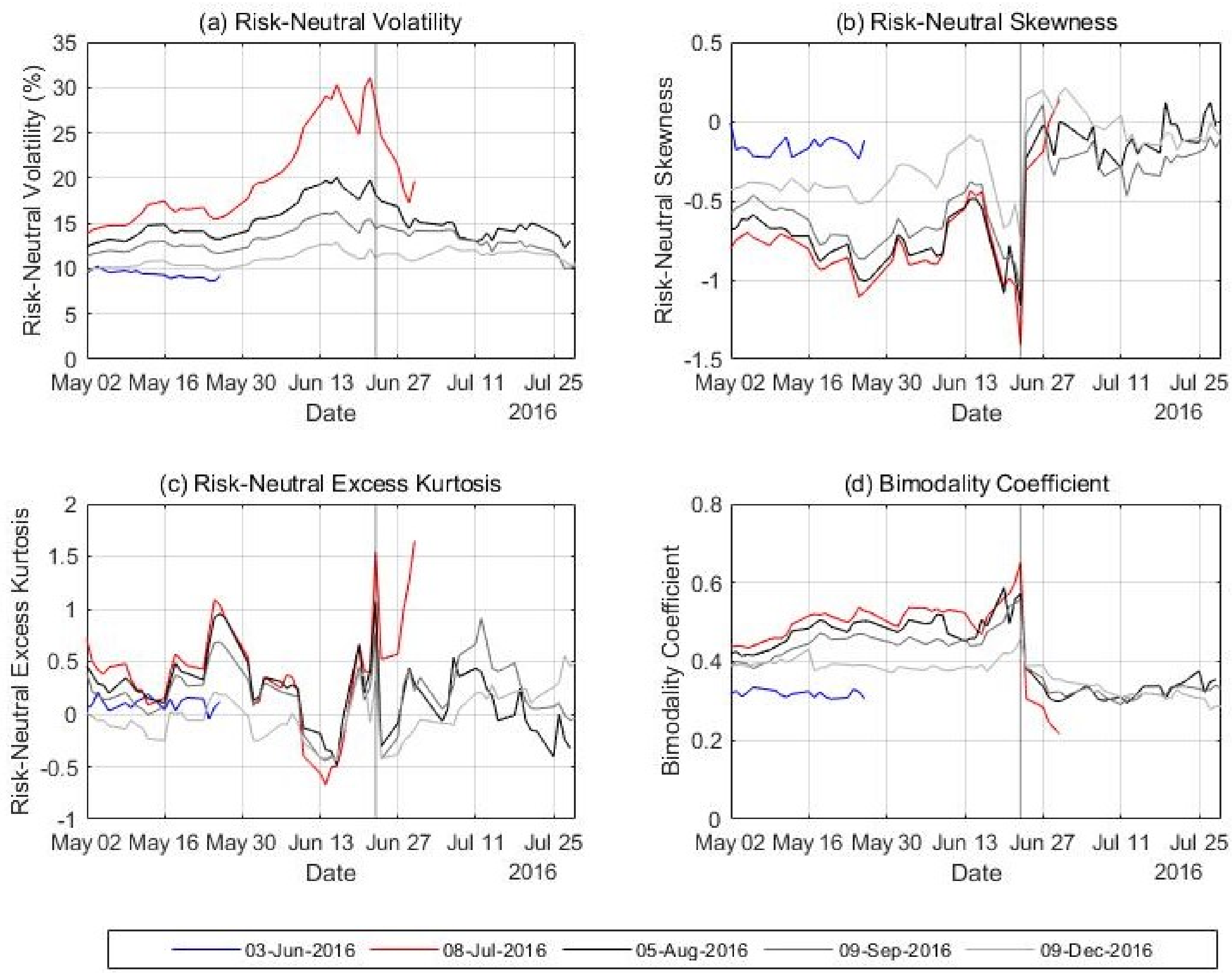
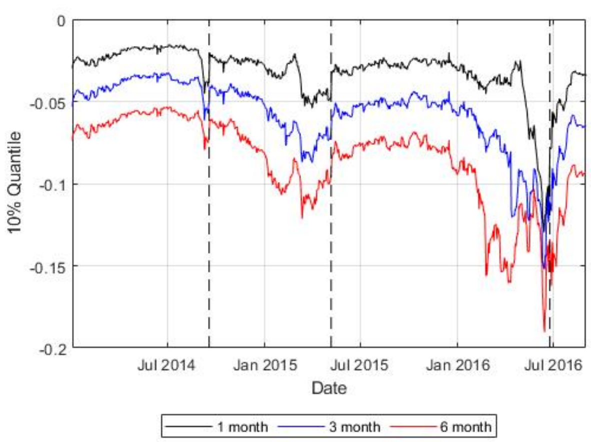


Figure 6: British Pound Risk-Neutral 10% Quantiles

This figure presents the British Pound Risk-Neutral 10% Quantiles for three horizons. The black, blue and red lines correspond to the variation of 10% quantile for the one-month, three-month and six-month horizons, respectively. Figure 6a display the 10% quantiles during the period between 2013 and 2016. Three vertical lines indicates Scottish independence referendum on 18 September 2014, United Kingdom general election on 7 May, 2015 and Brexit referendum on 23 June, 2016. In Fugure 6b, we zoom into the Brexit period.The vertical line shows the day of the Brexit referendum on 23 June, 2016.

(a) Period from 2013 to 2016



(b) Brexit Period

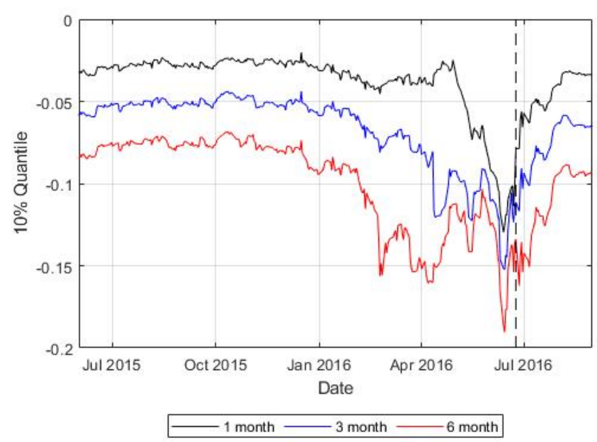


Figure 7: Risk-Neutral Brexit Probabilities Implied by Option and Prediction Markets

This figure presents the risk-neutral Brexit probabilities implied by option and prediction markets. The red line is the probability of Leave implied in the prediction market (Betfair) and the other three lines are risk-neutral probabilities of Leave implied in option market. Specifically, solid black, dash blue and dotted blue lines represent risk-neutral probabilities estimated from three different model specifications by setting $\omega_{t,i} = 1$, $\omega_{t,i} = \frac{1}{M_{t,K_i}}$ and $\omega_{t,i} = \frac{1}{M_{t,K_i}^2}$, respectively. The sample period used in this figure is from 3 May, 2016 to 23 June, 2016. The vertical lines show 14 and 16 June, 2016.

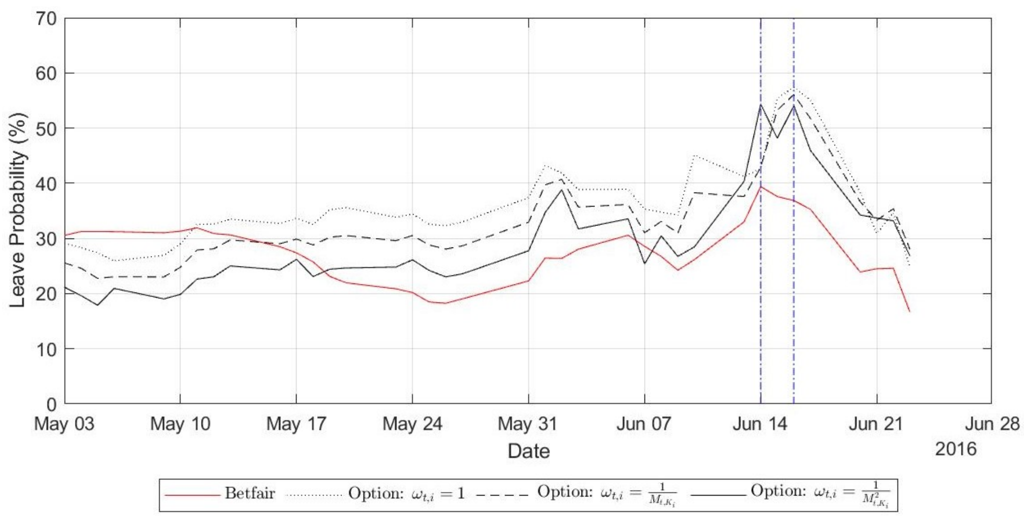
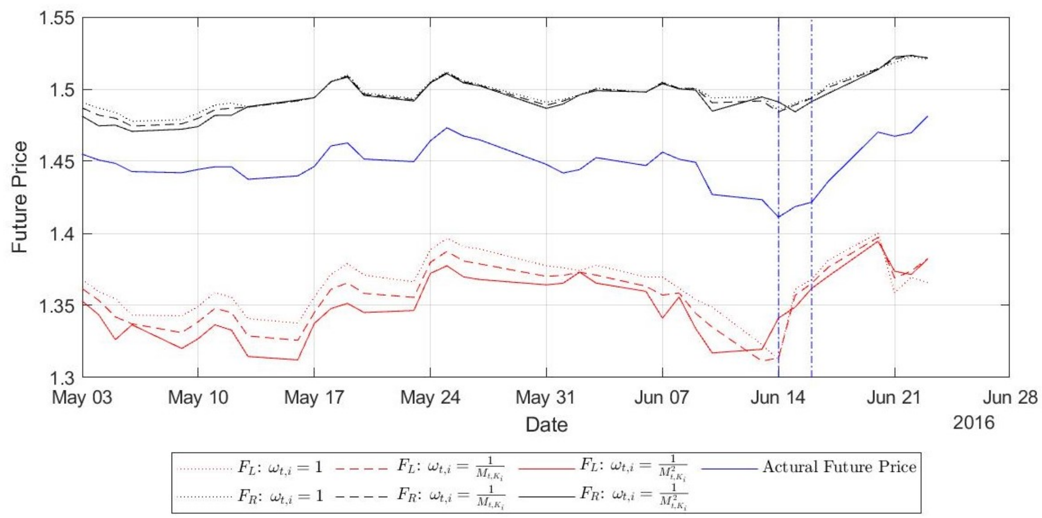


Figure 8: State Prices and Volatility of GBP Futures

This figure presents British Pound futures prices (Figure 8a) and corresponding volatilities (Figure 8b) in "Leave" and 'Remain' states. Blue line in Figure 8a is the actual British Pound futures price. Black and red lines in Figure 8a and Figure 8b represent British pound futures prices and corresponding volatilities in "Leave" and "Remain" state, receptively. Solid, dashed and dotted lines show state variables estimated from three different model specifications by setting $\omega_{t,i} = 1$, $\omega_{t,i} = \frac{1}{M_{t,K_i}}$ and $\omega_{t,i} = \frac{1}{M_{t,K_i}^2}$, respectively. The vertical lines show 14 and 16 June, 2016. The sample period used in this figure is from May 3, 2016 to June 23, 2016.

(a) Option Implied State Prices



(b) Option Implied State Volatility

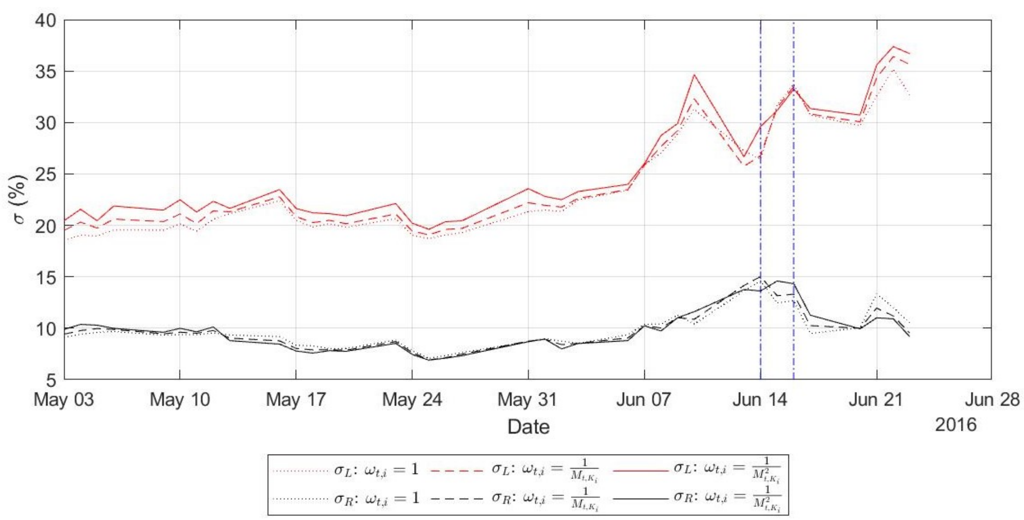


Figure 9: Leave Probabilities Recovered from the Model-free Method

This figure presents estimates of the "Leave" probability using a model-free method. Red circles represent the upper bounds of the estimated probability using the corresponding strike prices only. The black solid line is the fitted spline with 20 knots. The minimum of the spline is our preferred upper bound for the "Leave" probability. The sample period used in this estimation runs from the 2 May, 2016 and the 29 July, 2016.The dashed vertical line is the USD/GBP spot rate on the day of Brexit referendum, 23 June, 2016.

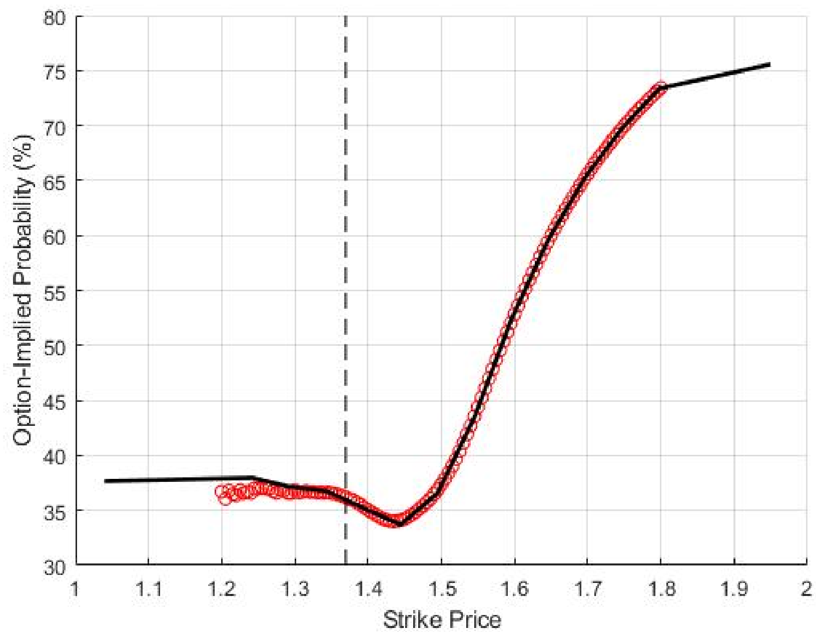


Figure 10: Leave Probabilities from the Political Opinion Polls

This figure presents the implied probability of "Leave" obtained from polls data (blue line), Betfair odds (red line) and option market (black line). The vertical lines show 14 June, 2016 and 16 June, 2016. The sample period is between 6 May, 2016 and 22 June, 2016.

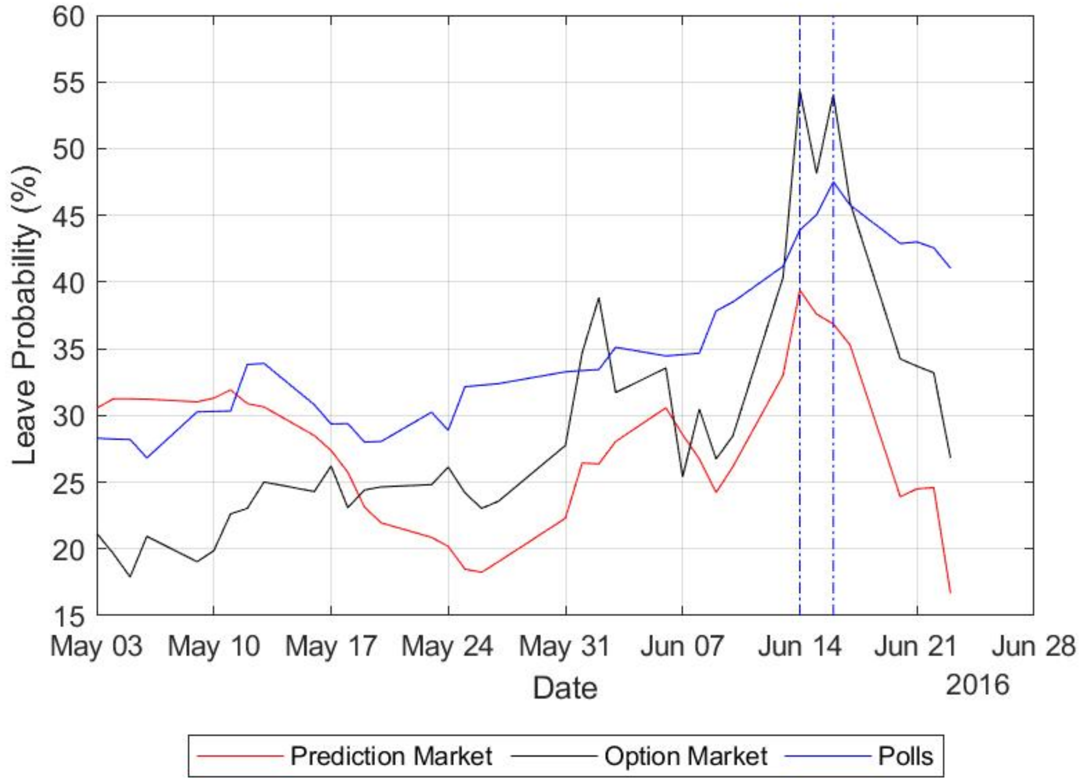
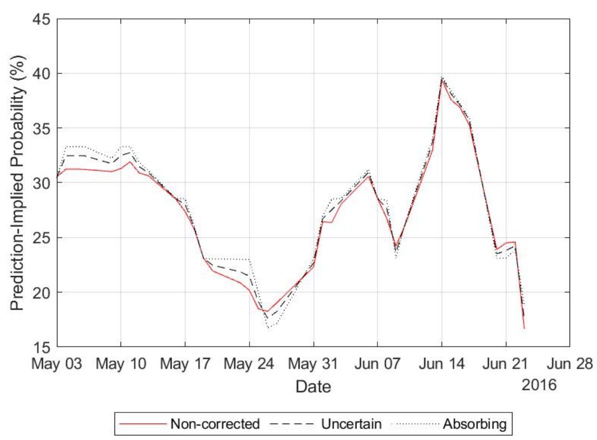


Figure 11: Risk-Corrected Leave Probability (Non-Parametric)

This figure presents the probabilities recovered using Ross (2015). Figure 11a and Figure 11b shows the non-corrected (risk-neutral) and risk-corrected probabilities in prediction markets and option markets, respectively. Red solid line is the non-corrected probabilities. Black dashed line is the risk-corrected probabilities corresponding to an (almost) absorbing state 2, where the Arrow-Debreu prices are $A_{21} = 0.01$ and $A_{22} = 0.99$. Black dotted line is the risk-corrected probabilities corresponding to "high uncertainty" state 2, where the Arrow-Debreu prices are $A_{21} = 0.5$ and $A_{22} = 0.5$. The sample period is between 3 May, 2016 and 23 June, 2016.

(a) Prediction Markets



(b) Option Markets

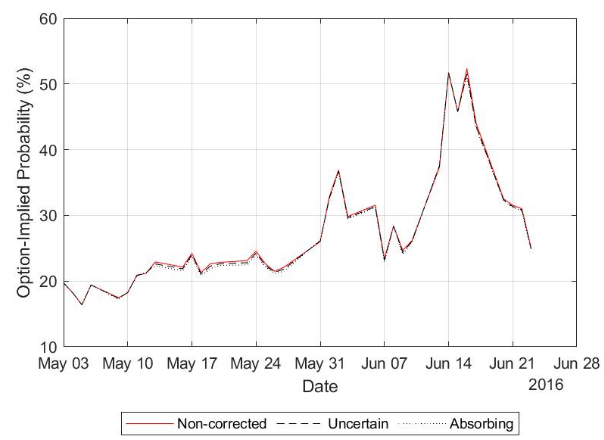
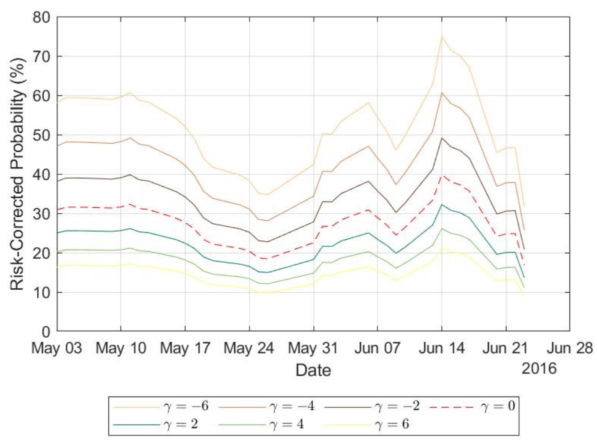


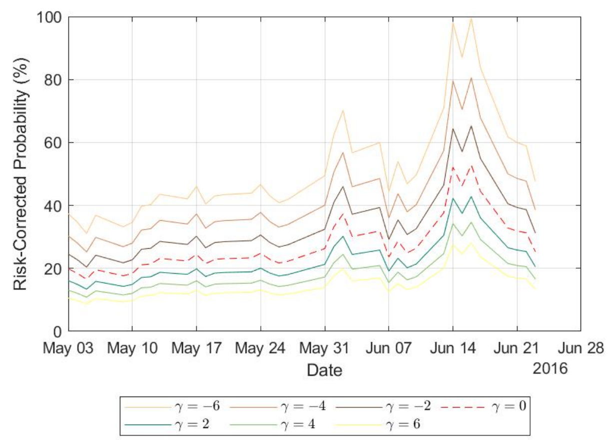
Figure 12: Risk-Corrected Leave Probability (Parametric)

This figure presents the probabilities recovered using the parametric approach. Figure 12a and 12c show the risk-corrected probabilities in prediction markets, and Figure 12b and 12d show the risk-corrected probabilities in option markets. In Figure 12a and 12b, assume the relative wealth in "Remain" and "Leave" states, $\frac{W^L}{W_R}$, is 0.90, we display the probabilities corrected using different value for the constant relative risk aversion coefficient ($\gamma = -6, -4, -2, 0, 2, 4, 6$). In Figure 12c and 12d, assume the constant relative risk aversion coefficient, γ, is -2, we display the probabilities corrected using different value for the relative wealth in "Remain" and "Leave" states ($\frac{W^L}{W_R} = 0.75, 0.90, 0.95, 1.00, 1.05, 1.10, 1.25$), where W^R is the wealth of the representative agent in the remain state and W^L the corresponding wealth in the leave state. The sample period is between 3 May, 2016 and 23 June, 2016.

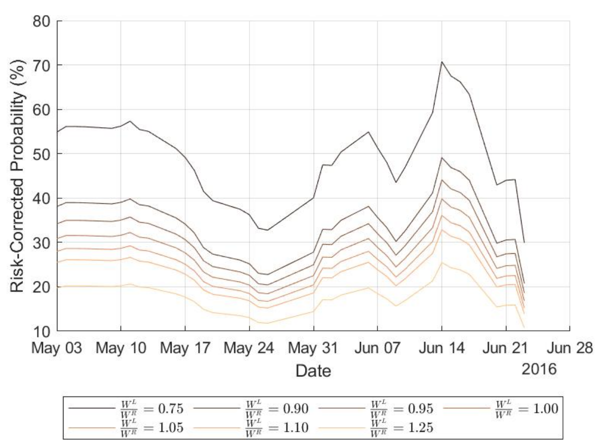
(a) Prediction Markets (γ)



(b) Option Markets (γ)



(c) Prediction Markets (Relative Wealth)



(d) Option Markets (Relative Wealth)

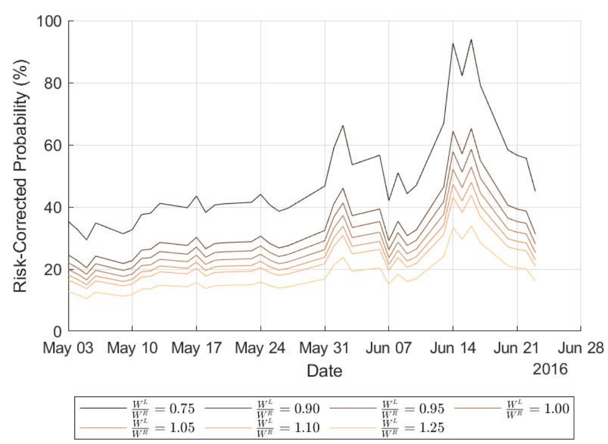
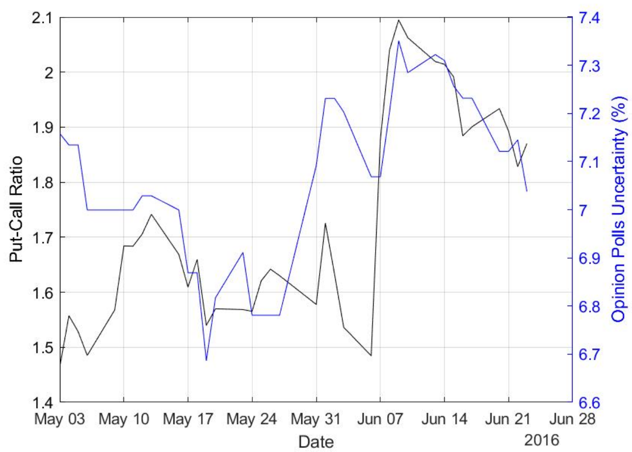


Figure 13: Trading Behaviors on Derivative Markets

This figure presents the trading behaviors on CME British pounds derivative markets. Figure 13a shows the put–call ratio (left axis) and the uncertainty of opinion polls outcomes (right axis). Put–call ratio is calculated from the open interest of CME British pounds options expiring on 8 July, 2016. The uncertainty of opinion polls outcomes is defined as the standard deviation of the expected "Leave" outcome under the assumption of binomial distribution, which is $\sqrt{\frac{p_{poll,t}^{L}(1-p_{poll,t})^{L}}{n}}$, and n is window length used for subjective "Leave" probability ($p_{poll,t}$) calculation. Figure 13b presents the total CME British pounds futures' and options' positions held by speculators (non-commercial) relative to the total positions held by hedgers (commercial), scaled by the values in the past period. The sample period in Figure 13a is between 3 May, 2016 and 23 June, 2016. The sample period in Figure 13b is between 3 May, 2016 and 28 June, 2016. The data is sampled weekly (Tuesday-to-Tuesday).

(a) Put Call Ratio and Polls Uncertainty



(b) Speculative vs. Hedging Activity

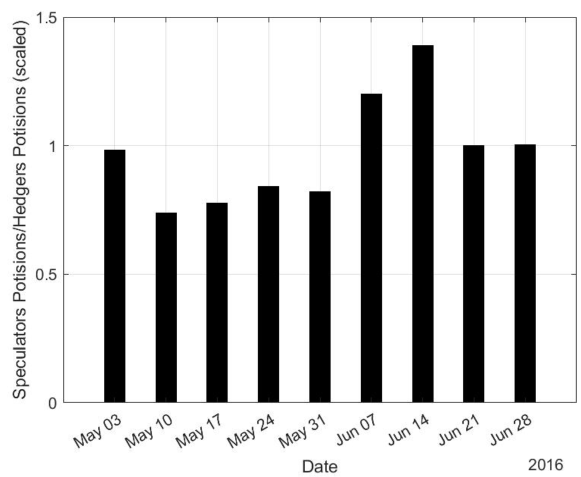


Figure 14: Risk-Corrected versus Subjective Probabilities

This figure presents the subjective probabilities extracted from opinion polls and risk-corrected probabilities by using data from the betting and option markets. Probabilities are corrected using the following parameterisation: (1) $\beta = 0.99$, (2) $\frac{W^L}{W^R} = 0.90$ where W^R is the level of wealth of the representative agent in the remain state and W^L the corresponding level of wealth in the leave state, (3)relative risk reversion coefficient, γ^{p*} and γ^{o*}, where p and o represent prediction and option markets respectively. Specifically, $\gamma^{i*} \equiv \arg\min_\gamma \sum_{t=1}^{T} (\tilde{\pi}_t^i - \hat{\pi}_t)^2$, where $i = p, o$, T is the sample size, $\tilde{\pi}_t^i$ is the risk-corrected probability by using date from option or prediction markets, and $\hat{\pi}_t$ is the subjective probabilities inferred from opinion polls; this gives $\gamma^{p*} = -1.83$ and $\gamma^{o*} = -1.01$.The sample period is between 6 May, 2016 and 22 June, 2016.

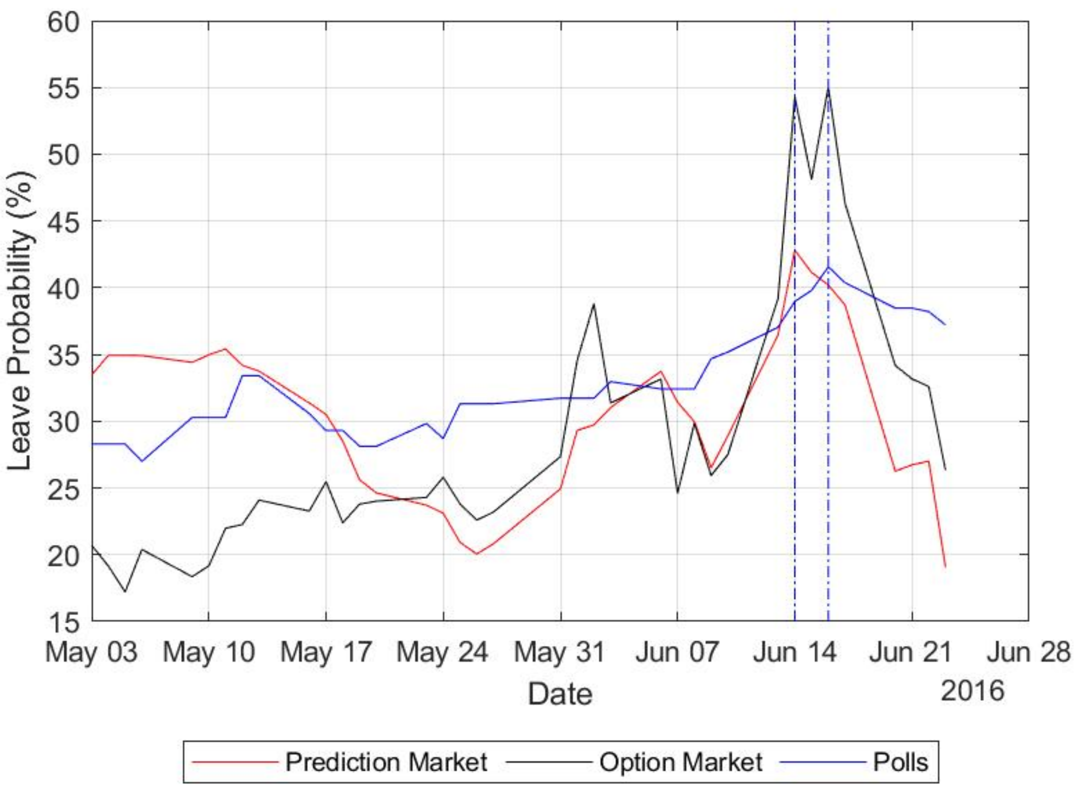


Figure 15: Relative Risk Aversion Coefficient and Relative Wealth

This figure presents the relative risk aversion coefficient used to minimize the distance between the risk-corrected probabilities and the physical probability from the BES data, with respect to different level of wealth in the "Leave" and "Remain" states. We use $\beta = 0.99$ and $\frac{W^L}{W^R}$ between 0.5 and 1.5, where W^R is the level of wealth of the representative agent in the "Remain" state and W^L the corresponding level of wealth in the "Leave" state. The red and black solid lines are the relative risk aversion coefficients for prediction and option markets, respectively. The dashed blue line represents the baseline case used in Figure 14. The sample period is between 6 May, 2016 and 22 June, 2016.

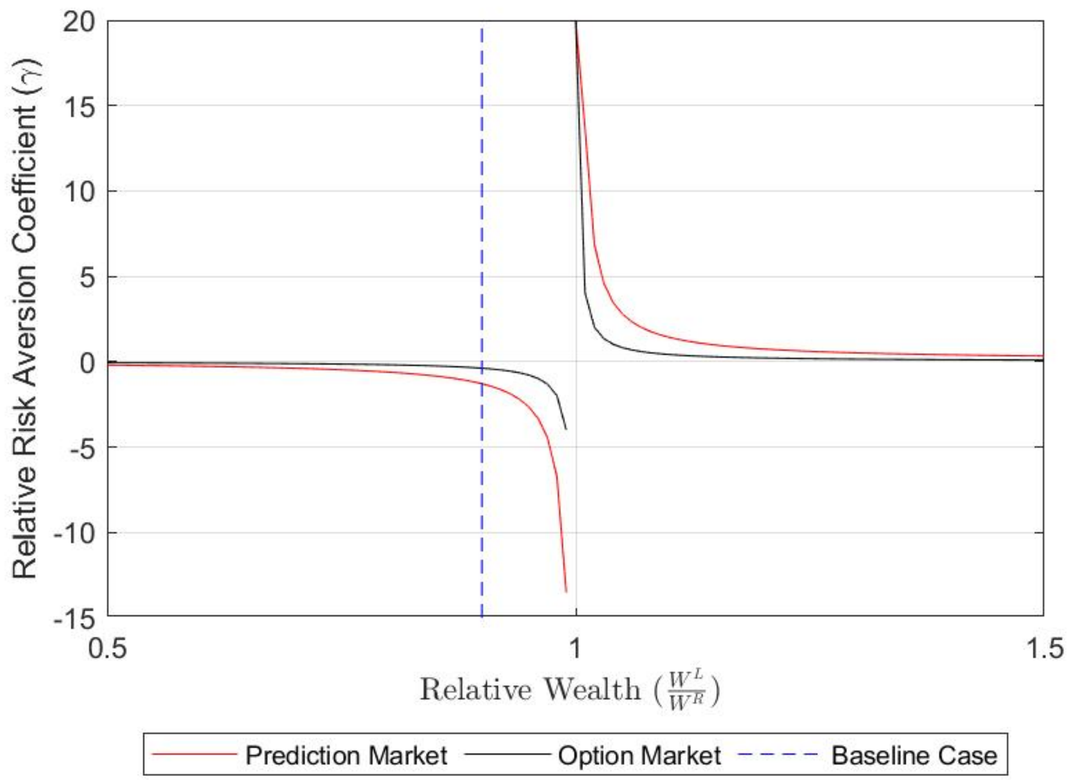


Figure 16: Voting Intention from the BES Survey

This figure presents the voting intention of ”Leave” from the BES Survey (blue line, right y-axis) and implied probability of “Leave” obtained from Betfair odds (red line, left y-axis) and option market (black line, left y-axis). The vertical lines show 14 June, 2016 and 16 June, 2016. The sample period is between 6 May, 2016 and 22 June, 2016.

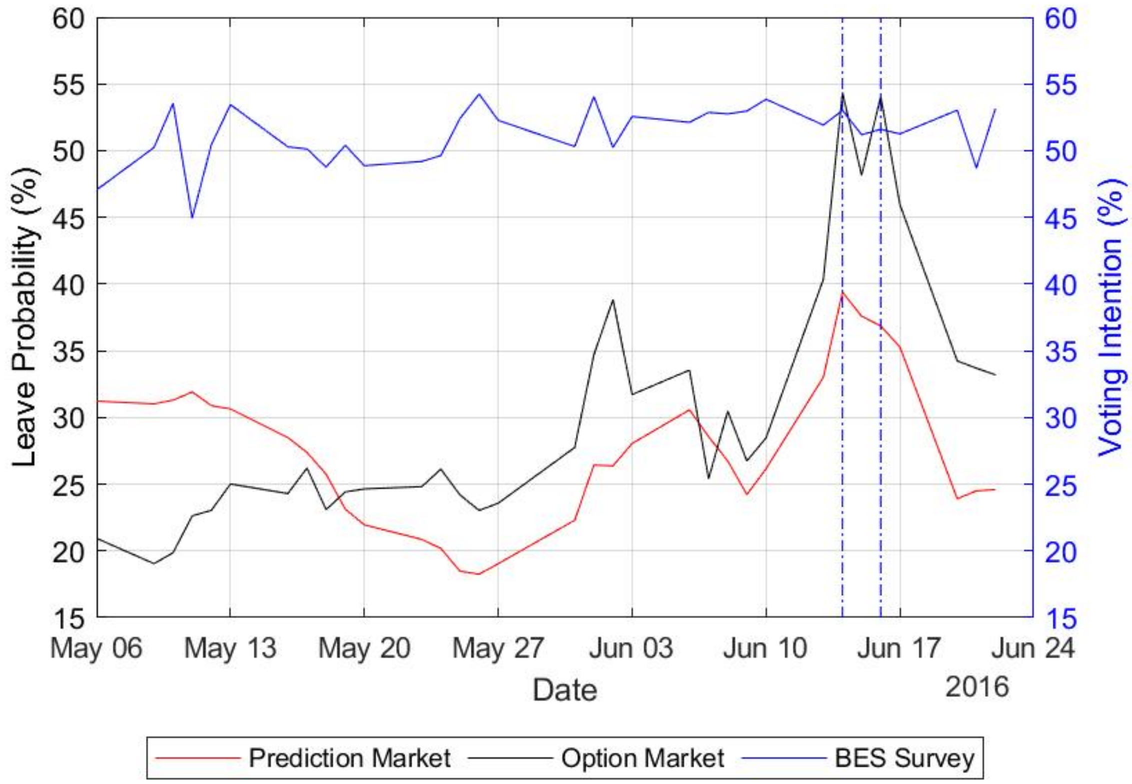


Table 1: Variables and Corresponding BES Questions

Variables	BES Question and Variables Values
Risk Taking	Generally speaking, how willing are you to take risks? 0 - ‘Very unwilling’ to 3 - ‘Very willing’
Female	Are you male or female?
Personal Annula Gross Income $\geq$ 20k	Gross PERSONAL income is an individual’s total income received from all sources, including wages, salaries, or rents and before tax deductions...What is your gross personal income?
Education	At what age did you finish full-time education?
Age	What is your age?
Party Identification	Generally speaking, do you think of yourself as Labour, Conservative, Liberal Democrat or what?
Level of Immigration Increasing	Do you think that each of the following are getting higher, getting lower or staying about the same?(The level of immigration) 0 - ‘Getting a lot lower’ to 4 - ‘Getting a lot higher’
Brexit Would Reduce Immigration	Do you think the following would be higher, lower or about the same if the UK leaves the European Union? (Immigration to the UK) 0 - ‘Much higher’ to 4 - ‘Much lower’
Immigrants Burden on Welfare State	How much do you agree or disagree with the following statements? (Immigrants are a burden on the welfare state) 0 - ‘Strongly disagree’ to 4 - ‘Strongly agree’
Immigration Bad for Economy	Do you think immigration is good or bad for Britain’s economy? 0 - ‘Good for economy’ to 6 - ‘Bad for economy’
Immigration Undermines Cultural Life	And do you think that immigration undermines or enriches Britain’s cultural life? 0 - ‘Enriches for Britain’s cultural life’ to 6 - ‘Undermines for Britain’s cultural life’
Eurosceptic Newspaper Reader	Which daily newspaper do you read most often? Eurosceptic newspaper are The Express, The Daily Mail / The Scottish Daily Mail, The Sun, The Daily Telegraph, and The Times.
British Identity	Where would you place yourself on these scales? 0 - ‘Not at all British’ to 6 - ‘Very strongly British’
English Identity	Where would you place yourself on these scales? 0 - ‘Not at all English’ to 6 - ‘Very strongly English’
European Identity	Where would you place yourself on these scales? 0 - ‘Not at all European to 6 - ‘Very strongly European’
Economic Pessimism	Do you think that each of the following are getting better, getting worse or staying about the same? (The economy) 0 - ‘Getting a lot better’ to 4 - ‘Getting a lot worse’

Table 2: Probit Regression Models of EU Referendum Vote Choice

This table presents result of the Probit regressions under five model specifications, where the dependent variable is the self-reported preferred choice at the 2016 referendum. T-statistics are reported in the parentheses. The sample period is between 6 May, 2016 and 22 June, 2016.

Variable	**(1)**		**(2)**		**(3)**		**(4)**		**(5)**	
Risk Taking Preference	0.11	(10.77)	0.19	(17.94)	0.15	(13.65)	0.24	(17.99)	0.27	(18.78)
Female			-0.01	(-0.82)	0.01	(0.60)	-0.04	(-2.21)	-0.10	(-4.61)
Personal Annula Gross Income $\geq$ 20k			-0.18	(-11.44)	-0.20	(-11.74)	-0.13	(-6.72)	-0.12	(-5.78)
Education (ref: Left School After 18):										
Left School Before 16			0.81	(44.76)	0.75	(38.81)	0.31	(13.11)	0.19	(7.39)
Left School Before 17 and 18			0.51	(26.80)	0.44	(21.88)	0.18	(7.49)	0.12	(4.70)
Age (ref: Aged 18-34)										
Aged 35-54			0.35	(15.17)	0.37	(15.15)	0.19	(6.81)	0.20	(6.69)
Aged $\geq$ 55			0.46	(20.90)	0.46	(19.32)	0.28	(10.13)	0.32	(10.59)
Party Identification (ref: Other/None)										
Conservative					0.29	(13.44)	0.01	(0.29)	-0.10	(-3.51)
Labour					-0.55	(-25.93)	-0.36	(-14.28)	-0.35	(-12.63)
Liberal Democrat					-0.66	(-19.19)	-0.37	(-9.21)	-0.27	(-6.35)
Nationalist					-0.52	(-13.62)	-0.36	(-7.77)	-0.08	(-1.46)
UKIP					1.80	(30.84)	1.22	(17.55)	1.10	(14.62)
Personal View about Immigration:										
Level of Immigration Increasing							0.30	(24.54)	0.25	(18.60)
Brexit Would Reduce Immigration							0.42	(37.56)	0.38	(32.17)
Immigrants Burden on Welfare State							0.21	(19.11)	0.16	(13.49)
Immigration Bad for Economy							0.12	(13.99)	0.08	(8.31)
Immigration Undermines Cultural Life							0.14	(19.22)	0.09	(11.54)
Eurosceptic Newspaper Reader									0.25	(11.20)
Identity:										
British Identity									0.02	(2.89)
English Identity									0.07	(13.60)
European Identity									-0.31	(-50.21)
Economic Pessimism									0.09	(7.54)
Constant	-0.14	(-8.42)	-0.94	(-31.91)	-0.80	(-24.48)	-4.01	(-69.12)	-3.15	(-42.00)
Adj. Generalized R^2	0.50%		16.03%		32.53%		61.09%		67.99%	
N	31008		31008		31008		31008		31008	
Share of Leave Voters	50.93%		50.93%		50.93%		50.93%		50.93%	
Intention to Leave (excluding Indecisive Respondents)	50.93%		50.94%		50.87%		50.94%		50.95%	
Intention to Leave of Indecisive Respondents	50.11%		50.92%		50.16%		53.58%		52.85%	
Intention to Leave (Total)	50.88%		50.94%		50.83%		51.10%		51.07%	

Table 3: Correlation between the Change of Physical and Risk-Neutral Probabilities

This table presents the correlations between the change of physical probabilities and the change of risk-neutral probabilities from option and prediction markets, display in column "Correlation (with Option)" and "Correlation (with Prediction)", respectively. Physical probabilities are inferred from political opinion polls by using 18 different learning models, including two methods (expending and rolling windows), three type of weights (equal-weighted, exponential decay and linear decay), three window lengths (30, 60 and 90 days). The sample period is between 3 May, 2016 and 23 June, 2016.

Method	**Weight**	**Windows (days)**	**Correlation (with Option)**	**Correlation (with Prediction)**
Expanding	**Equal-weighted**	**30**	0.16	0.29
Expanding	**Exponential Decay**	**30**	0.10	0.24
Expanding	**Linear Decay**	**30**	0.13	0.28
Expanding	**Equal-weighted**	**60**	0.21	0.33
Expanding	**Exponential Decay**	**60**	0.17	0.31
Expanding	**Linear Decay**	**60**	0.20	0.33
Expanding	**Equal-weighted**	**90**	0.24	0.36
Expanding	**Exponential Decay**	**90**	0.21	0.34
Expanding	**Linear Decay**	**90**	0.23	0.36
Rolling	**Equal-weighted**	**30**	0.16	0.18
Rolling	**Exponential Decay**	**30**	0.16	0.30
Rolling	**Linear Decay**	**30**	0.24	0.39
Rolling	**Equal-weighted**	**60**	0.20	0.34
Rolling	**Exponential Decay**	**60**	0.22	0.36
Rolling	**Linear Decay**	**60**	0.29	0.42
Rolling	**Equal-weighted**	**90**	0.25	0.34
Rolling	**Exponential Decay**	**90**	0.26	0.39
Rolling	**Linear Decay**	**90**	**0.30**	**0.43**
	Min		0.10	0.18
	Max		0.30	0.43
	Average		0.21	0.33
	Standard Deviation		0.05	0.06

Appendix for:

Risk-Corrected Probabilities of a Binary Event

Appendix A: RNDs Extraction

First, we only use options with less noise and more informative prices for the RND extraction. Our selection is mainly based on options' trading volume and the open interest. Specifically, due to the thin trading volume of in-the-money (ITM) options and deep-out-of-the-money (deep OTM) options, we exclude them from our sample. We classify options with a price equal to \$ 0.001 per pound and a price change less than \$ 0.001 per pound increment as deep OTM. In addition, we discard options with zero open interest. Thus, we only use out-of-the-money (OTM) and at-the-money (ATM) options with positive open interest for the RND extraction.

Second, we covert option prices from price-strike space to volatility-strike space. Since options in our sample are American-style options written on futures, we use the Barone-Adesi-Whaley (BAW) American futures option pricing model (Barone-Adesi and Whaley, 1987) to obtain the implied volatility, which can eliminate the early exercise premium in the American option. Cincibuch (2004) studies the difference between the implied volatilities of CME American-style Japanese Yen futures options and OTC European-style Japanese Yen options. They find that there is no significant difference between BAW implied volatilities from American options and Black–Scholes (BS) implied volatilities from European options. Therefore, we use BAW implied volatilities to infer implied volatilities of European options and follow the standard procedure for extracting RND from European options.

Third, we smooth implied volatility curves. Previous studies adopt various methods to smooth implied volatility curves, for example, quadratic polynomial (Shimko, 1993) and cubic splines (Bates, 1991; Jiang and Tian, 2005; Monteiro et al., 2008). Due to the nature of the quadratic polynomial, the quadratic spline is not continuous in the second derivative. A standard cubic spline can avoid this issue, but it must pass the original data points, which brings noise from market microstructure frictions. To avoid these issues, we fit the implied volatility curve by using a fourth-degree polynomial spline (Figlewski, 2009), which minimizes the sum of squared differences between the observed implied volatility curve and the fitted spline. Hence, implied volatilities can be interpolated as a dense set of fitted splines.

Fourth, we convert interpolated implied volatilities back to option prices. Because of the insignificant difference between BAW implied volatility and implied volatility from its corresponding European option, we use the interpolated BAW implied volatility in the Black model (Black, 1976) to calculate the price of the corresponding European option.

In the fifth step, we approximate the middle part of the RND by using interpolated European futures options prices. The middle part of RND at time t is between the second lowest and the second highest observed strike prices at time t. Let F_{t,T_f} denote the price of a future contract with maturity T_f at time t. Assume that T_o is an option's expiry date, then the time-T_o payoff of an European call option written on a future contract, with strike price K, is written $\max(F_{T_o,T_f} - K, 0)$. We denote $C(t,K,T_f,T_o)$ as the observed call option price at time t, with strike price K, maturity T_o and underlying future contract that expires at T_f. With the assumption of no arbitrage, the option price at time t is equal to the present value of the risk-neutral expected payoff at T_o,

$$\begin{aligned} C(t,K,T_f,T_o) &= e^{-r_t(T_o-t)} \mathbb{E}_t[\max(F_{T_o,T_f} - K, 0)] \\ &= e^{-r_t(T_o-t)} \int_K^\infty (F_{T_o,T_f} - K)\pi_t(F_{T_o,T_f})\, dF_{T_o,T_f}, \end{aligned} \tag{A.28}$$

where r_t is the risk-free interest rate at time t, $\pi_t(F_{T_o,T_f})$ is the risk-neutral probability density function of the underlying future price. Following the standard approach of Breeden and Litzenberger (1978),

$$\frac{\partial C(t,K,T_f,T_o)}{\partial K} = e^{-r_t(T_o-t)}\left[\int_0^K \pi_t(F_{T_o,T_f})\, dF_{T_o,T_f} - 1\right]. \tag{A.29}$$

Thus, the risk-neutral cumulative function of the underlying future price at time t is

$$\Pi_t(K) = Prob_t(F_{T_o,T_f} \leq K) = \int_0^K \pi_t(F_{T_o,T_f})\, dF_{T_o,T_f} = 1 + e^{r_t(T_o-t)} \frac{\partial C(t,K,T_f,T_o)}{\partial K}. \tag{A.30}$$

Taking the derivative with respect to K in equation (A.30), the risk-neutral probability density function is

$$\pi_t(K) = e^{r_t(T_o - t)} \frac{\partial^2 C(t, K, T_f, T_o)}{\partial K^2}. \tag{A.31}$$

Taking finite differences, we approximate the option-implied risk-neutral cumulative density function (CDF)

$$\Pi_t(K) \approx 1 + e^{r_t(T_o - t)} \frac{1}{\triangle}[C(t, K + \frac{\triangle}{2}, T_f, T_o) - C(t, K - \frac{\triangle}{2}, T_f, T_o)] \mid_{\triangle \to 0}, \tag{A.32}$$

and risk-neutral probability density function (PDF), which is also known as RND,

$$\pi_t(K) \approx e^{r_t(T_o - t)} \frac{1}{\triangle^2}[C(t, K + \triangle, T_f, T_o) + C(t, K - \triangle, T_f, T_o) - 2C(t, K, T_f, T_o)] \mid_{\triangle \to 0}. \tag{A.33}$$

Similar as the steps we extract RND from a set of call option prices, we can obtain the risk-neutral CDF from put option prices as the following,

$$\Pi_t(K) \approx e^{r_t(T_o - t)} \frac{1}{\triangle}[P(t, K + \frac{\triangle}{2}, T_f, T_o) - P(t, K - \frac{\triangle}{2}, T_f, T_o)] \mid_{\triangle \to 0}, \tag{A.34}$$

and its corresponding risk-neutral probability density function (PDF)

$$\pi_t(K) \approx e^{r_t(T_o - t)} \frac{1}{\triangle^2}[P(t, K + \triangle, T_f, T_o) + P(t, K - \triangle, T_f, T_o) - 2P(t, K, T_f, T_o)] \mid_{\triangle \to 0}. \tag{A.35}$$

Then, $\pi_t(K)$ is the middle part of the RND. Finally, we append the RND into the left and right tails by fitting the generalized extreme value (GEV) distribution, which is a natural candidate to model the tails of an unknown distribution. GEV distribution has three parameters and its CDF is

$$\Pi_{GEV}(z) = e^{-(1+\xi z)^{-\frac{1}{\xi}}}, z = \frac{F_{t,T_1} - \mu}{\sigma}, \tag{A.36}$$

where ξ, μ and σ are used to control shape, location and scale of the tail distribution. $\xi > 0$ indicates a fat tail from the Frechet distribution, $\xi = 0$ determines a normal tail with the Gumbel distribution, and $\xi < 0$ means finite tails from the Weibull distribution. We define the strike price at α-quantile of the RND as $K(\alpha)$, which is equivalent to $\Pi_{GEV}(K(\alpha)) = \alpha$. $\Pi_{GEV}(z)$'s corresponding PDF can be written as $\pi_{GEV}(z)$. To estimate three parameters in GEV distribution, we set up three constraints for the right tail as the following:

$$\begin{aligned}
&\Pi_{GEV}(K(\alpha_{1R})) = \alpha_{1R}, \\
&\pi_{GEV}(K(\alpha_{1R})) = \pi(K(\alpha_{1R})), \\
&\pi_{GEV}(K(\alpha_{2R})) = \pi(K(\alpha_{2R})),
\end{aligned} \tag{A.37}$$

where $\pi(K(\alpha))$ is the empirical RND function we estimated in the last step. The constraints used for left tail estimation are

$$\begin{aligned}
&\Pi_{GEV}(-K(\alpha_{1L})) = 1 - \alpha_{1L}, \\
&\pi_{GEV}(-K(\alpha_{1L})) = \pi(K(\alpha_{1L})), \\
&\pi_{GEV}(-K(\alpha_{2L})) = \pi(K(\alpha_{2L})).
\end{aligned} \tag{A.38}$$

Specifically, we set α_1 and α_2 as 5% and 2% for the left tail, and 92% and 95% for the right tail.

Appendix B: AD Prices for Risk Recovery

B.1 AD Prices from Betting Odds

Consider the first observation in our sample. The timestamp is *25/02/2016 16:28:00*. Odds are 3.2£ for leave and 1.5£ for remain. The corresponding Arrow-Debrew prices for these states of nature are $\frac{1}{3.2}£$ and $\frac{1}{1.5}£$. This means that we can write

$$\mathbf{A_t} = \begin{pmatrix} \frac{1}{1.5} & \frac{1}{3.2} \\ A_{21,t} & A_{22,t} \end{pmatrix}. \tag{B.39}$$

We will use two assumptions to complete the second row of $\mathbf{A_t}$. The first assumption is of "high uncertainty" as embedded in the following AD prices

$$\mathbf{A_t} = \begin{pmatrix} \frac{1}{1.5} & \frac{1}{3.2} \\ 0.5 & 0.5 \end{pmatrix}, \tag{B.40}$$

and second, an (almost) absorbing state 2

$$\mathbf{A_t} = \begin{pmatrix} \frac{1}{1.5} & \frac{1}{3.2} \\ 0 & 1 \end{pmatrix}. \tag{B.41}$$

Note that, under risk neutrality and no discounting, the Arrow-Debrew pricing matrix is equivalent to the frequency matrix, i.e. $\mathbf{A_t} = \mathbf{P_t}$.

B.2 AD Prices from FX Derivatives

The option price as well as the forward price in the left hand side of the system of equations (5) is expressed in USD. In other words, those contracts refer to the dollar price of a future pound (whatever state of nature, i.e leave or remain, realises). Hence, p_t^L refers to the probability that the GBP will reach a certain *dollar* value in each and respective state of nature. In order to calculate primitive or state prices we need the domestic (GBP) price of a future unit of domestic currency (GBP) in each state of nature. Let us maintain the hypothesis that there are only two states: leave or remain. The time t GBP price of the AD securities are given by A_t^{R*} and A_t^{L*} whereas the time t USD price of the AD securities are written as A_t^R and A_t^L, for the remain and leave states, respectively. Calculating the asset return from the perspective of a US resident and assuming no-arbitrage results

$$\underbrace{\frac{1}{A_t^R + A_t^L}}_{\text{USD } T_f \text{ return of investing one USD at } t} = \underbrace{\underbrace{\frac{\overbrace{S_t^{-1}}^{\text{one USD in GBP}}}{A_t^{R*} + A_t^{L*}}}_{\text{GBP } T_f \text{ return of investing one USD at } t} \times F_{t,T_f}}_{\text{USD } T_f \text{ return of investing one USD at } t}, \tag{B.42}$$

or

$$A_t^{R*} + A_t^{L*} = \left(A_t^R + A_t^L\right)\frac{F_{t,T_f}}{S_t}. \tag{B.43}$$

Note also that $A_t^R + A_t^L = \frac{1}{1+i_t}$ and $A_t^{R*} + A_t^{L*} = \frac{1}{1+i_t^*}$, where i_t is the risk-free interest rate on a US bond which matures at T_f; the asterisk refers to the one in the UK economy. This gives

$$F_{t,T_f} = (1+i_t)S_t\left(A_t^{R*} + A_t^{L*}\right). \tag{B.44}$$

Using the notation in the first section, recall that F_{t,T_f} is the USD price at time t of one unit of a future time, T_f, GBP. An agent that buys F_{t,T_f} will receive one GBP at T_f for certain, i.e. irrespective of the state of nature. It follows that $\frac{F_{t,T_f}}{S_t}$ is the GBP price at time t of one unit of a future GBP. This transformation is equivalent to the price of a risk free asset which can be divided into two primitive (theoretical) sterling prices, at t. The first is given by $\left(1-p_t^L\right) \times \frac{F_{t,T_f}^L}{S_t}$ which can be seen as an asset that will pay one unit of GBP if (and only if) the leave state of nature realises, i.e., if $p_t^L = 0$. The second is $p_t^L \times \frac{F_{t,T_f}^R}{S_t}$ which has an analogous interpretation, however, for the remain case. Given that S_t is the dollar price of a spot GBP and F_t is the dollar price of a future GBP, one can write (B.44) as

$$p_t^L F_{t,T_f}^L + (1-p_t^L)F_{t,T_f}^R = (1+i_t)S_t\left(A_t^{L*} + A_t^{R*}\right), \tag{B.45}$$

or

$$A_t^{L*} + A_t^{R*} = \underbrace{p_t^L \frac{F_{t,T_f}^L}{S_t(1+i_t)}}_{\text{which equals } A_t^{L*} \text{ by arbitrage}} + \underbrace{(1-p_t^L)\frac{F_{t,T_f}^R}{S_t(1+i_t)}}_{\text{which equals } A_t^{R*} \text{ by arbitrage}} . \tag{B.46}$$

Appendix C: An Example for Non-parametric Recovery

Consider the symmetric case of AD prices in the betting market, in other words,

$$\mathbf{A_t} = \begin{pmatrix} \frac{1}{1.5} & \frac{1}{3.2} \\ \frac{1}{3.2} & \frac{1}{1.5} \end{pmatrix}. \tag{C.47}$$

Eigenvalues are $\phi_1 = 0.9791667$ and $\phi_2 = 0.3541667$; the corresponding eigenvectors are $Z_{1,t} = (0.7071068\ ,\ 0.7071068)^\top$ and $Z_{2,t} = (-0.7071068\ ,\ 0.7071068)^\top$. PF theorem guarantees that for square non-negative matrices, ϕ_1 is the highest in absolute value and all entries of the corresponding eigenvector are positive. This allow us to assume that $\phi_1 = \beta$ and

$$\mathbf{D_t} = \begin{pmatrix} 0.7071068 & 0 \\ 0 & 0.7071068 \end{pmatrix}, \tag{C.48}$$

which finally gives

$$\mathbf{P_t} = \begin{pmatrix} 0.6808511 & 0.3191489 \\ 0.3191489 & 0.6808511 \end{pmatrix}, \tag{C.49}$$